

Real and Virtual Compton Scattering at Low Energies*

S. Scherer

Institut für Kernphysik, Johannes Gutenberg-Universität, D-55099 Mainz, Germany

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Abstract

These lectures give a pedagogical introduction to real and virtual Compton scattering at low energies. We will first discuss real Compton scattering off a point particle as well as a composite system in the framework of nonrelativistic quantum mechanics. The concept of electromagnetic polarizabilities is introduced. We then address a description of the Compton-scattering tensor within quantum field theory with particular emphasis on the derivation of low-energy theorems. The importance of a consistent treatment of hadron structure in the use of electromagnetic vertices is stressed. Finally, the reader is introduced to the notion of generalized polarizabilities in the rapidly expanding field of virtual Compton scattering.

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I. INTRODUCTION

The discovery of the Compton effect [1,2], *i.e.*, the scattering of photons off electrons, and its explanation in terms of conservation of energy and momentum in the collision between a single light quantum with an electron is regarded as one of the key developments of modern physics [3]. In atomic physics, condensed matter physics, and chemistry Compton scattering is nowadays an important tool of investigating the momentum distribution of the scattering electrons in the probe. The inclusion of the electron spin in the calculation of the Compton-scattering cross section by Klein and Nishina [4] has become one of the textbook examples of applying quantum electrodynamics at lowest order.

In the realm of strong-interaction physics, the potential of using Compton scattering as a method of studying properties of particles was realized in the early fifties. The influence of the *anomalous* magnetic moment of the proton on the Compton-scattering cross section was first discussed by Powell [5]. The derivation of low-energy theorems (LETs), *i.e.*, model-independent predictions based upon a few general principles, became an important starting point in understanding hadron structure [6–9]. Typically, the leading terms of the low-energy amplitude for a given reaction are predicted in terms of global, model-independent properties of the particles. LETs provide an important constraint for models or theories of hadron structure: unless these general principles are violated, the predictions of a low-energy theorem must be reproduced. Furthermore, LETs also provide useful constraints for experiments as they define a reference point for the precision which has to be achieved in experimental studies designed to distinguish between different models.

Based on the requirement of gauge invariance, Lorentz invariance, crossing symmetry, and the discrete symmetries, the low-energy theorem for Compton scattering (CS) of real photons off a nucleon [8,9] uniquely specifies the terms in the low-energy scattering amplitude up to and including terms linear in the photon momentum. The coefficients of this expansion are expressed in terms of global properties of the nucleon: its mass, charge, and magnetic moment. Terms of second order in the frequency, which are not determined by this theorem, can be parameterized in terms of two new structure constants, the electric and magnetic polarizabilities of the nucleon. These polarizabilities have been the subject of numerous experimental and theoretical studies as they determine the first information on the compositeness or structure of the nucleon specific to Compton scattering.

As in all studies with electromagnetic probes, the possibilities to investigate the structure of the target are much greater if virtual photons are used, since the energy and three-momentum of the virtual photon can be varied independently. Moreover, the longitudinal component of current operators entering the amplitude can be studied. The amplitude for virtual Compton scattering (VCS) off the proton is accessible in the reactions $e^-p \rightarrow e^-p\gamma$ and $\gamma p \rightarrow pe^-e^+$. In particular, the first process has recently received considerable interest as it allows to investigate generalizations of the RCS polarizabilities to the spacelike region, namely, the so-called generalized polarizabilities [10].

The purpose of these lectures is to provide an *introduction* to the topics of real and virtual Compton scattering. The material is organized in three chapters. We start at an elementary level in the framework of nonrelativistic quantum mechanics and discuss basic features of Compton scattering. Then, a covariant treatment within quantum field theory is discussed with particular emphasis on a consistent treatment of compositeness in the use

of electromagnetic vertices. In the last chapter, the reader is introduced to the rapidly expanding field of virtual Compton scattering.

In preparing these lectures, we have made use of the excellent pedagogical reviews on hadron polarizabilities of Refs. [11–13]. A vast amount of more detailed information is contained in Refs. [14,15]. An overview of the current status of experimental and theoretical activities on hadron polarizabilities can be found in Ref. [16]. Finally, for a first review on virtual Compton scattering the reader is referred to Ref. [17].

II. COMPTON SCATTERING IN NONRELATIVISTIC QUANTUM MECHANICS

A. Kinematics and notations

We will first discuss real Compton scattering (RCS), for which $q^2 = q'^2 = q \cdot \epsilon = q' \cdot \epsilon' = 0$. The kinematical variables and polarization vectors are defined in Fig. 1. As a result of translational invariance in space-time, the total three-momentum and energy, respectively, are conserved,

$$\vec{p}_i + \vec{q} = \vec{p}_f + \vec{q}', \quad E_i + \omega = E_f + \omega', \quad (1)$$

where the energy of the particle is given by $E = \frac{\vec{p}^2}{2M}$ or $E = \sqrt{M^2 + \vec{p}^2}$ depending on whether one uses a nonrelativistic or relativistic framework. For the description of the RCS amplitude one requires two kinematical variables, *e.g.*, the energy of the initial photon, ω , and the scattering angle between the initial photon and the scattered photon, $\cos(\Theta) = \hat{q} \cdot \hat{q}'$. The energy of the scattered photon in the lab frame is given by

$$\omega' = \frac{\omega}{1 + \frac{\omega}{M}[1 - \cos(\Theta)]}, \quad (2)$$

if use of relativistic kinematics is made. From Eq. (2) one obtains the well-known result for the wavelength shift of the Compton effect, $\Delta\lambda = (4\pi/M) \sin^2(\Theta/2)$.

B. Nonrelativistic Compton scattering off a point particle

In order to set the stage, we will first discuss, in quite some detail, Compton scattering of real photons off a free point particle of mass M and charge $e > 0$ within the framework of nonrelativistic quantum mechanics. First of all, this will allow us to introduce basic concepts such as gauge invariance, photon-crossing symmetry as well as discrete symmetries. Secondly, the result will define a reference point beyond which the structure of a composite object can be studied. Finally, this will also allow us to discuss later on, where a relativistic description departs from a nonrelativistic treatment.

Consider the Hamiltonian of a single, free point particle of mass M and charge $e > 0$,¹

¹Except for a few cases, we will use the same symbols for quantum-mechanical operators such as $\hat{\vec{p}}$ and corresponding eigenvalues \vec{p} .

$$H_0 = \frac{\vec{p}^2}{2M}. \quad (3)$$

The coupling to the electromagnetic field, $A^\mu(\vec{x}, t) = (\Phi(\vec{x}, t), \vec{A}(\vec{x}, t))$, is generated by the well-known minimal-substitution procedure²

$$i\frac{\partial}{\partial t} \mapsto i\frac{\partial}{\partial t} - e\Phi(\vec{x}, t), \quad \vec{p} \mapsto \vec{p} - e\vec{A}(\vec{x}, t), \quad (4)$$

resulting in the Schrödinger equation

$$i\frac{\partial\Psi(\vec{x}, t)}{\partial t} = [H_0 + H_I(t)]\Psi(\vec{x}, t) = [H_0 + H_1(t) + H_2(t)]\Psi(\vec{x}, t) = H(\Phi, \vec{A})\Psi(\vec{x}, t), \quad (5)$$

where

$$H_1(t) = -e\frac{\vec{p} \cdot \vec{A} + \vec{A} \cdot \vec{p}}{2M} + e\Phi, \quad H_2(t) = \frac{e^2}{2M}\vec{A}^2. \quad (6)$$

Gauge invariance of Eq. (5) means that

$$\Psi'(\vec{x}, t) = \exp[-ie\chi(\vec{x}, t)]\Psi(\vec{x}, t) \quad (7)$$

is a solution of

$$i\frac{\partial\Psi'(\vec{x}, t)}{\partial t} = H(\Phi + \dot{\chi}, \vec{A} - \vec{\nabla}\chi)\Psi'(\vec{x}, t), \quad (8)$$

provided $\Psi(\vec{x}, t)$ is a solution of Eq. (5). In other words, Eq. (5) remains invariant under a gauge transformation

$$\Psi \mapsto \exp[-ie\chi(\vec{x}, t)]\Psi, \quad A^\mu \mapsto A^\mu + \partial^\mu\chi. \quad (9)$$

For a discussion of gauge invariance in the context of nonrelativistic reductions the interested reader is referred to Ref. [18].

After introducing the interaction representation,

$$H_I^{int}(t) = e^{iH_0 t} H_I(t) e^{-iH_0 t}, \quad (10)$$

the S -matrix element is obtained by evaluating the Dyson series

$$S = 1 + \sum_{k=1}^{\infty} \frac{(-i)^k}{k!} \int_{-\infty}^{\infty} dt_1 \cdots dt_k \hat{T} [H_I^{int}(t_1) \cdots H_I^{int}(t_k)] \quad (11)$$

between $|i\rangle \equiv |\vec{p}_i; \gamma(q, \epsilon)\rangle$ and $\langle f| \equiv \langle \vec{p}_f; \gamma(q', \epsilon')|$. In Eq. (11), \hat{T} refers to the time-ordering operator,

²We use Heaviside-Lorentz units, $e > 0$, $\alpha = e^2/4\pi \approx 1/137$.

$$\hat{T}[A(t_1)B(t_2)] = A(t_1)B(t_2)\Theta(t_1 - t_2) + B(t_2)A(t_1)\Theta(t_2 - t_1), \quad (12)$$

with a straightforward generalization to an arbitrary number of operators. We use second-quantized photon fields

$$\langle 0|A^\mu(\vec{x}, t)|\gamma[q, \epsilon(\lambda)]\rangle = \epsilon^\mu(q, \lambda)N(\omega)e^{-iq \cdot x}, \quad (13)$$

where $N(\omega) = [(2\pi)^3 2\omega]^{-1/2}$, and normalize the states as

$$\langle \vec{x}|\vec{p}\rangle = \frac{e^{i\vec{p} \cdot \vec{x}}}{\sqrt{(2\pi)^3}}. \quad (14)$$

The part relevant to Compton scattering $[\mathcal{O}(e^2)]$ reads

$$S = -i \int_{-\infty}^{\infty} dt H_2^{int}(t) - \int_{-\infty}^{\infty} dt_1 dt_2 H_1^{int}(t_1) H_1^{int}(t_2) \Theta(t_1 - t_2), \quad (15)$$

where the first term generates the contact-interaction contribution or so-called seagull term:

$$\begin{aligned} S_{fi}^{cont} &= -i \frac{e^2}{2M} \int_{-\infty}^{\infty} dt \langle f|e^{iH_0 t} \vec{A}^2(\vec{r}, t) e^{-iH_0 t}|i\rangle \\ &= -i(2\pi)^4 \delta^4(p_f + q' - p_i - q) \frac{1}{\sqrt{4\omega\omega'}(2\pi)^6} \frac{e^2 \vec{\epsilon}'^* \cdot \vec{\epsilon}}{M}. \end{aligned} \quad (16)$$

In order to obtain Eq. (16), one first contracts the photon field operators with the photons in the initial and final states, respectively,³ then evaluates the time integral, and, finally, makes use of

$$\langle \vec{p}'|f(\vec{r})|\vec{p}\rangle = \frac{1}{(2\pi)^3} \int d^3 r e^{i(\vec{p}-\vec{p}') \cdot \vec{r}} f(\vec{r})$$

with $f(\vec{r}) = \exp[i(\vec{q} - \vec{q}') \cdot \vec{r}]$, to obtain the three-momentum conservation. The second contribution of Eq. (15) is evaluated by inserting a complete set of states between $H_1^{int}(t_1)$ and $H_1^{int}(t_2)$:

$$- \int_{-\infty}^{\infty} dt_1 dt_2 \Theta(t_1 - t_2) \int d^3 p \langle f|H_1^{int}(t_1)|\vec{p}\rangle \langle \vec{p}|H_1^{int}(t_2)|i\rangle. \quad (17)$$

There are two distinct possibilities to contract the photon fields, namely, $A_\nu(t_2)$ with $|\gamma(q, \epsilon)\rangle$ and $A_\mu(t_1)$ with $\langle \gamma(q', \epsilon')|$ and vice versa, giving rise to the so-called direct and crossed channels, respectively. Evaluating the time dependence and making use of

$$\int_{-\infty}^{\infty} dt_1 dt_2 \Theta(t_1 - t_2) e^{iat_1} e^{ibt_2} = \frac{2\pi i \delta(a+b)}{a + i0^+}$$

one obtains

³Note the factor of 2 for two contractions.

$$S_{fi}^{dc+cc} = -2\pi i \delta(E_f + \omega' - E_i - \omega) \int d^3p \left(\frac{\langle \vec{p}_f | H_1^{em} | \vec{p} \rangle \langle \vec{p} | H_1^{abs} | \vec{p}_i \rangle}{E_f + \omega' - E(\vec{p}) + i0^+} + \frac{\langle \vec{p}_f | H_1^{abs} | \vec{p} \rangle \langle \vec{p} | H_1^{em} | \vec{p}_i \rangle}{E_f - \omega - E(\vec{p}) + i0^+} \right), \quad (18)$$

where the superscripts *abs* and *em* refer to absorption and emission of photons, respectively, and where the matrix elements are given by

$$\langle \vec{p} | H_1^{abs} | \vec{p}_i \rangle = -eN(\omega) \delta^3(\vec{p} - \vec{q} - \vec{p}_i) \left[\frac{(\vec{p} + \vec{p}_i) \cdot \vec{\epsilon}}{2M} - \epsilon_0 \right], \quad (19)$$

$$\langle \vec{p}_f | H_1^{em} | \vec{p} \rangle = -eN(\omega') \delta^3(\vec{p}_f + \vec{q}' - \vec{p}) \left[\frac{(\vec{p}_f + \vec{p}) \cdot \vec{\epsilon}'^*}{2M} - \epsilon_0'^* \right]. \quad (20)$$

Using the following convention

$$\mathcal{T}_{fi} = \frac{1}{\sqrt{4\omega\omega'}(2\pi)^6} t_{fi},$$

with $S = I + iT$ at the operator level and $\langle f | T | i \rangle = (2\pi)^4 \delta^4(P_f - P_i) \mathcal{T}_{fi}$, the final result for the T -matrix element reads

$$t_{fi} = e^2 \left\{ -\frac{\vec{\epsilon}'^* \cdot \vec{\epsilon}}{M} - \left[\frac{(2\vec{p}_f + \vec{q}') \cdot \vec{\epsilon}'^*}{2M} - \epsilon_0'^* \right] \frac{1}{E_f + \omega' - E(\vec{p}_f + \vec{q}')} \left[\frac{(2\vec{p}_i + \vec{q}) \cdot \vec{\epsilon}}{2M} - \epsilon_0 \right] - \left[\frac{(2\vec{p}_f - \vec{q}) \cdot \vec{\epsilon}}{2M} - \epsilon_0 \right] \frac{1}{E_f - \omega - E(\vec{p}_f - \vec{q})} \left[\frac{(2\vec{p}_i - \vec{q}') \cdot \vec{\epsilon}'^*}{2M} - \epsilon_0'^* \right] \right\}. \quad (21)$$

Let us discuss a few properties of t_{fi} .

- Gauge invariance: As a result of the gauge-invariance property of the equation of motion, the result for true observables should not depend on the gauge chosen. In the present context, this means that the transition matrix element is invariant under the replacement $\epsilon^\mu \rightarrow \epsilon^\mu + \zeta q^\mu$ (analogously for ϵ'):

$$\begin{aligned} t_{fi} &\xrightarrow{\epsilon^\mu \rightarrow q^\mu} e^2 \left\{ -\frac{\vec{\epsilon}'^* \cdot \vec{q}}{M} - \left[\frac{(2\vec{p}_f + \vec{q}') \cdot \vec{\epsilon}'^*}{2M} - \epsilon_0'^* \right] \frac{1}{E_f + \omega' - E(\vec{p}_f + \vec{q}')} \left[\frac{(2\vec{p}_i + \vec{q}) \cdot \vec{q}}{2M} - \omega \right] - \left[\frac{(2\vec{p}_f - \vec{q}) \cdot \vec{q}}{2M} - \omega \right] \frac{1}{E_f - \omega - E(\vec{p}_f - \vec{q})} \left[\frac{(2\vec{p}_i - \vec{q}') \cdot \vec{\epsilon}'^*}{2M} - \epsilon_0'^* \right] \right\} \\ &\stackrel{(*)}{=} e^2 \left\{ -\frac{\vec{\epsilon}'^* \cdot \vec{q}}{M} + \left[\frac{(2\vec{p}_f + \vec{q}') \cdot \vec{\epsilon}'^*}{2M} - \epsilon_0'^* \right] - \left[\frac{(2\vec{p}_i - \vec{q}') \cdot \vec{\epsilon}'^*}{2M} - \epsilon_0'^* \right] \right\} \\ &= 0 \quad \text{since} \quad 2\vec{p}_f + \vec{q}' - 2\vec{p}_i + \vec{q}' = 2\vec{q}, \end{aligned}$$

where, using energy conservation, in $(*)$ we inserted

$$\begin{aligned} E_f + \omega' - E(\vec{p}_f + \vec{q}') &= - \left[\frac{(2\vec{p}_i + \vec{q}) \cdot \vec{q}}{2M} - \omega \right], \\ E_f - \omega - E(\vec{p}_f - \vec{q}) &= \left[\frac{(2\vec{p}_f - \vec{q}) \cdot \vec{q}}{2M} - \omega \right]. \end{aligned}$$

- Photon-crossing symmetry: t_{fi} is invariant under the simultaneous replacements $\epsilon^\mu \leftrightarrow \epsilon'^{\mu*}$ and $q^\mu \leftrightarrow -q'^\mu$, *i.e.*,

$$t_{fi} \mapsto e^2 \left\{ -\frac{\vec{\epsilon} \cdot \vec{\epsilon}'^*}{M} - \left[\frac{(2\vec{p}_f - \vec{q}) \cdot \vec{\epsilon}}{2M} - \epsilon_0 \right] \frac{1}{E_f - \omega - E(\vec{p}_f - \vec{q})} \left[\frac{(2\vec{p}_i - \vec{q}') \cdot \vec{\epsilon}'^*}{2M} - \epsilon_0'^* \right] \right. \\ \left. - \left[\frac{(2\vec{p}_f + \vec{q}') \cdot \vec{\epsilon}'^*}{2M} - \epsilon_0'^* \right] \frac{1}{E_f + \omega' - E(\vec{p}_f + \vec{q}')} \left[\frac{(2\vec{p}_i + \vec{q}) \cdot \vec{\epsilon}}{2M} - \epsilon_0 \right] \right\} \\ = t_{fi}.$$

- Invariance under $e \mapsto -e$, *i.e.*, the Compton-scattering amplitudes for particles of charges e and $-e$ are identical.
- Under parity, t_{fi} behaves as a scalar, *i.e.*, there are no terms of, *e.g.*, the type $\epsilon_{ijk}\epsilon_i\epsilon_j^*q_k$.
- Particle crossing, $(E_i, \vec{p}_i) \leftrightarrow (-E_f, -\vec{p}_f)$, is *not* a symmetry of a nonrelativistic treatment.

For the purpose of calculating the differential cross section, we make use of the Coulomb gauge, $\epsilon_0 = 0$, $\vec{q} \cdot \vec{\epsilon} = 0$, $\epsilon_0' = 0$, and $\vec{q}' \cdot \vec{\epsilon}' = 0$, and evaluate the S -matrix element in the laboratory frame, where $\vec{p}_i = 0$,

$$S_{fi} = -i(2\pi)^4 \delta(E_f + \omega' - E_i - \omega) \delta^3(\vec{p}_f + \vec{q}' - \vec{p}_i - \vec{q}) \frac{1}{\sqrt{4\omega\omega'}(2\pi)^6} \frac{e^2 \vec{\epsilon} \cdot \vec{\epsilon}'^*}{M}.$$

Using standard techniques,⁴ the differential cross section reads

$$d\sigma = \delta(E_f + \omega' - E_i - \omega) \delta^3(\vec{p}_f + \vec{q}' - \vec{p}_i - \vec{q}) \frac{1}{\omega\omega'} \left| \frac{e^2 \vec{\epsilon} \cdot \vec{\epsilon}'^*}{4\pi M} \right|^2 d^3q' d^3p_f.$$

After integration with respect to the momentum of the particle, we make use of $d^3q' = \omega'^2 d\omega' d\Omega$ and obtain

$$\frac{d\sigma}{d\Omega} = \left\{ 1 - 4 \frac{\omega}{M} \sin^2 \left(\frac{\Theta}{2} \right) + \mathcal{O} \left[\left(\frac{\omega}{M} \right)^2 \right] \right\} \left| \frac{e^2 \vec{\epsilon} \cdot \vec{\epsilon}'^*}{4\pi M} \right|^2. \quad (22)$$

Averaging and summing over initial and final photon polarizations, respectively, is easily performed by treating $\{\hat{q} = \hat{e}_z, \vec{\epsilon}(1) = \hat{e}_x, \vec{\epsilon}(2) = \hat{e}_y\}$ as well as $\{\hat{q}', \vec{\epsilon}'(1), \vec{\epsilon}'(2)\}$ as orthonormal bases,

$$\sum_{\lambda'=1}^2 \left(\frac{1}{2} \sum_{\lambda=1}^2 |\vec{\epsilon}(\lambda) \cdot \vec{\epsilon}'^*(\lambda')|^2 \right) = \frac{1}{2} [1 + \cos^2(\Theta)]. \quad (23)$$

⁴It is advantageous to discuss these steps using box normalization instead of δ -function normalization. See Chap. 7.11 of Ref. [19].

Let us consider the so-called Thomson limit, *i.e.*, $\omega \rightarrow 0$, for which Eq. (22) in combination with Eq. (23) reduces to

$$\left. \frac{d\sigma}{d\Omega} \right|_{\omega=0} = \frac{\alpha^2}{M^2} \frac{1 + \cos^2(\Theta)}{2}, \quad \alpha = \frac{e^2}{4\pi} \approx \frac{1}{137}.$$

The total cross section, obtained by integrating over the entire solid angle, reproduces the classical Thomson scattering cross section denoted by σ_T ,

$$\sigma_T = \frac{8\pi}{3} \frac{\alpha^2}{M^2}. \quad (24)$$

Numerical values of the Thomson cross section for the electron, charged pion, and the proton are shown in Table I.

C. Nonrelativistic Compton scattering off a composite system

Next we discuss Compton scattering off a composite system within the framework of nonrelativistic quantum mechanics. For the sake of simplicity, we consider a system of two particles interacting via a central potential $V(r)$,

$$H_0 = \frac{\vec{p}_1^2}{2m_1} + \frac{\vec{p}_2^2}{2m_2} + V(|\vec{r}_1 - \vec{r}_2|) = \frac{\vec{P}^2}{2M} + \frac{\vec{p}^2}{2\mu} + V(r), \quad (25)$$

where we introduced

$$M = m_1 + m_2, \quad \vec{R} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{M}, \quad \vec{P} = \vec{p}_1 + \vec{p}_2, \\ \mu = \frac{m_1 m_2}{M}, \quad \vec{r} = \vec{r}_1 - \vec{r}_2, \quad \vec{p} = \frac{m_2 \vec{p}_1 - m_1 \vec{p}_2}{M}.$$

As in the single-particle case, the electromagnetic interaction is introduced via minimal coupling, $i\partial/\partial t \rightarrow i\partial/\partial t - q_1\phi_1 - q_2\phi_2$, $\vec{p}_i \rightarrow \vec{p}_i - q_i \vec{A}_i$, resulting in the interaction Hamiltonians

$$H_1(t) = \sum_{i=1}^2 \left[-\frac{q_i}{2m_i} (\vec{p}_i \cdot \vec{A}_i + \vec{A}_i \cdot \vec{p}_i) + q_i \phi_i \right], \\ H_2(t) = \sum_{i=1}^2 \frac{q_i^2}{2m_i} \vec{A}_i^2,$$

where $(\phi_i, \vec{A}_i) = (\phi(\vec{r}_i, t), \vec{A}(\vec{r}_i, t))$. In order to keep the expressions as simple as possible, we will make some simplifying assumptions and quote the general result at the end. First of all, we do not consider the spin of the constituents, *i.e.*, we omit an interaction term

$$-\sum_i \vec{\mu}_i \cdot \vec{B}_i, \quad \vec{B}_i = \vec{\nabla}_i \times \vec{A}_i,$$

where $\vec{\mu}_i$ is an intrinsic magnetic dipole moment of the i th constituent.

Secondly, we take equal masses for the constituents, $m_1 = m_2 = m = \frac{1}{2}M$ and assume that one has charge $q_1 = e > 0$ and the second one is neutral, $q_2 = 0$. Finally, as a result of the gauge-invariance property we perform the calculation within the Coulomb gauge, $\phi_i = 0$.⁵ With these preliminaries, the Hamiltonian reads

$$H = H_0 - \frac{e}{M}(\vec{p}_1 \cdot \vec{A}_1 + \vec{A}_1 \cdot \vec{p}_1) + \frac{e^2}{M}\vec{A}_1^2.$$

The S -matrix element is obtained in complete analogy to the previous section within the framework of ordinary time-dependent perturbation theory:

$$S_{fi} = S_{fi}^{cont} + S_{fi}^{dc} + S_{fi}^{cc}, \quad (26)$$

where the seagull contribution results from the sum of the individual contact terms and the direct-channel and crossed-channel contributions are more complicated as in the single-particle case, since they now also involve excitations of the composite object.

Using $\vec{r}_1 = \vec{R} + \frac{1}{2}\vec{r}$, one obtains for the contact contribution

$$t_{fi}^{cont} = -\vec{\epsilon} \cdot \vec{\epsilon}'^* \frac{2e^2}{M} \int d^3r |\phi_0(\vec{r})|^2 \exp \left[i(\vec{q} - \vec{q}') \cdot \frac{\vec{r}}{2} \right]. \quad (27)$$

Since $q_2 = 0$, the integral is just the charge form factor $F[(\vec{q} - \vec{q}')^2]$ of the ground state,

$$F(\vec{q}^2) = 1 - \frac{1}{6}r_E^2 \vec{q}^2 + \dots$$

We note that for a composite object, in general, the contact interactions of the constituents do not yet generate the complete Thomson limit. However, it is possible to make a unitary transformation such that the total Thomson amplitude is generated by a contact term making the composite object look very similar to the point object [23].

The second contribution is evaluated by inserting a complete set of states,

$$S_{fi}^{dc+cc} = -2\pi i \delta(E_f + \omega' - E_i - \omega) \int d^3P \sum_n \times \left(\frac{\langle \vec{p}_f, 0 | H_1^{em} | \vec{P}, n \rangle \langle \vec{P}, n | H_1^{abs} | \vec{p}_i, 0 \rangle}{E_f + \omega' - E_n(\vec{P})} + \frac{\langle \vec{p}_f, 0 | H_1^{abs} | \vec{P}, n \rangle \langle \vec{P}, n | H_1^{em} | \vec{p}_i, 0 \rangle}{E_f - \omega - E_n(\vec{P})} \right), \quad (28)$$

where, in the framework of Eq. (25), the energy of an excited state with intrinsic energy ω_n moving with c.m. momentum \vec{P} is given by

$$E_n(\vec{P}) = \frac{\vec{P}^2}{2M} + \omega_n.$$

⁵ In actual calculations, it is highly recommended not to specify the gauge and use gauge invariance as a check of the final result.

In Coulomb gauge, the corresponding Hamiltonians for absorption and emission of photons, respectively, read

$$H_1^{abs} = -\frac{2e}{M}N(\omega)\hat{\vec{p}}_1 \cdot \vec{\epsilon} \exp(i\vec{q} \cdot \vec{r}_1), \quad H_1^{em} = -\frac{2e}{M}N(\omega')\hat{\vec{p}}_1 \cdot \vec{\epsilon}'^* \exp(-i\vec{q}' \cdot \vec{r}_1).$$

As in the point-object case, S_{fi} is symmetric under photon crossing $(\omega, \vec{q}) \leftrightarrow (-\omega', -\vec{q}')$ and $\vec{\epsilon} \leftrightarrow \vec{\epsilon}'^*$.

The low-energy expansion of Eq. (28) is obtained by expanding the vector potentials and the denominators in ω and ω' . The explicit calculation is beyond the scope of the present treatment and we will only quote the general result at the end [20–22]. However, we find it instructive to consider the limit $\omega \rightarrow 0$:

$$t_{fi}^{dc+cc}\big|_{\omega=0} = \frac{4e^2}{M^2} \sum_n \frac{1}{\Delta\omega_n} (<0|\vec{p} \cdot \vec{\epsilon}'^*|n><n|\vec{p} \cdot \vec{\epsilon}|0> + <0|\vec{p} \cdot \vec{\epsilon}|n><n|\vec{p} \cdot \vec{\epsilon}'^*|0>),$$

where $\Delta\omega_n = \omega_n - \omega_0$. The matrix elements involve internal degrees of freedom, only. Making use of $\vec{p} = i\mu[H_0, \vec{r}]$ and applying H_0 appropriately to the right or left, the expression simplifies to

$$\begin{aligned} t_{fi}^{dc+cc}\big|_{\omega=0} &= -i\frac{4e^2\mu}{M^2} \sum_n (<0|\vec{r} \cdot \vec{\epsilon}'^*|n><n|\vec{p} \cdot \vec{\epsilon}|0> - <0|\vec{p} \cdot \vec{\epsilon}|n><n|\vec{r} \cdot \vec{\epsilon}'^*|0>) \\ &= -i\frac{4e^2\mu}{M^2} <0|[\vec{r} \cdot \vec{\epsilon}'^*, \vec{p} \cdot \vec{\epsilon}]|0> = \frac{e^2}{M} \vec{\epsilon}'^* \cdot \vec{\epsilon}, \end{aligned} \quad (29)$$

where, again, we used the completeness relation, $[\vec{a} \cdot \hat{\vec{r}}, \vec{b} \cdot \hat{\vec{p}}] = i\vec{a} \cdot \vec{b}$, and $\mu = M/4$. Combining this result with the contact contribution of Eq. (27) yields the correct Thomson limit also for a composite system. Indeed, it has been shown a long time ago in the more general framework of quantum field theory that the scattering of photons in the limit of zero frequency is correctly described by the classical Thomson amplitude [6–9]. We will come back to this point in the next section.

Beyond the Thomson limit, we only quote the nonrelativistic T -matrix element for Compton scattering off a spin-zero particle of mass M and total charge Ze , expanded to second order in the photon energy:

$$t_{fi} = \vec{\epsilon}'^* \cdot \vec{\epsilon} \left(-\frac{(Ze)^2}{M} + 4\pi\bar{\alpha}_E\omega\omega' \right) + 4\pi\bar{\beta}_M\vec{q}' \times \vec{\epsilon}'^* \cdot \vec{q} \times \vec{\epsilon}, \quad (30)$$

where

$$\bar{\alpha}_E = \frac{\alpha Z^2 r_E^2}{3M} + 2\alpha \sum_{n \neq 0} \frac{|<n|D_z|0>|^2}{E_n - E_0}, \quad (31)$$

$$\bar{\beta}_M = -\frac{\alpha <\vec{D}^2>}{2M} - \frac{\alpha}{6} <\sum_{i=1}^N \frac{q_i^2 \vec{r}_i^2}{m_i}> + 2\alpha \sum_{n \neq 0} \frac{|<n|M_z|0>|^2}{E_n - E_0} \quad (32)$$

denote the electric ($\bar{\alpha}_E$) and magnetic ($\bar{\beta}_M$) polarizabilities of the system. In these equations

$$\vec{D} = \sum_{i=1}^N q_i (\vec{r}_i - \vec{R})$$

refers to the intrinsic electric dipole operator and

$$\vec{M} = \sum_{i=1}^N \left[\frac{\hat{q}_i}{2m_i} (\vec{r}_i - \vec{R}) \times (\vec{p}_i - \frac{m_i}{M} \vec{P}) + \vec{\mu}_i \right]$$

to the magnetic dipole operator, where the possibility of magnetic moments of the constituents has now been included. The electromagnetic polarizabilities describe the response of the internal degrees of freedom of a system to a small external electromagnetic perturbation. For example, in atomic physics the second term of Eq. (31) is related to the quadratic Stark effect describing the energy shift of an atom placed in an external electric field. We will come back to an interpretation of the electric polarizability in terms of a classical analogue in the next subsection.

Finally, let us discuss the influence of the electromagnetic polarizabilities on the differential Compton-scattering cross section. We restrict ourselves to the leading term due to the interference of the Thomson amplitude with the polarizability contribution. The evaluation of that term requires, in addition to Eq. (23), the sum

$$\sum_{\lambda, \lambda'} \text{Re} \{ \vec{\epsilon}'^* \cdot \vec{\epsilon} \hat{q}' \times \vec{\epsilon}' \cdot \hat{q} \times \vec{\epsilon}^* \} = 2 \cos(\Theta),$$

and one obtains

$$\begin{aligned} \frac{d\sigma}{d\Omega} = & \left[1 - 4 \frac{\omega}{M} \sin^2 \left(\frac{\Theta}{2} \right) + \mathcal{O} \left(\frac{\omega^2}{M^2} \right) \right] \left\{ \frac{1}{2} [1 + \cos^2(\Theta)] \frac{\alpha^2 Z^4}{M^2} \right. \\ & \left. - [1 + \cos^2(\Theta)] \frac{\alpha Z^2}{M} \bar{\alpha}_E \omega \omega' - 2 \cos(\Theta) \frac{\alpha Z^2}{M} \bar{\beta}_M \omega \omega' + \mathcal{O}(\omega^2 \omega'^2) \right\}. \end{aligned} \quad (33)$$

The differential cross sections at $\Theta = 0^\circ, 90^\circ$, and 180° are sensitive to $\bar{\alpha}_E + \bar{\beta}_M$, $\bar{\alpha}_E$, and $\bar{\alpha}_E - \bar{\beta}_M$, respectively.

D. Classical interpretation

The prototype example for illustrating the concept of an electric polarizability is a system of two harmonically bound point particles of mass m with opposite charges $+q$ and $-q$ [11,12]:

$$H_0 = \frac{\vec{p}_1^2}{2m} + \frac{\vec{p}_2^2}{2m} + \frac{\mu \omega_0^2}{2} \vec{r}^2, \quad \mu = \frac{m}{2}, \quad \vec{r} = \vec{r}_1 - \vec{r}_2,$$

where we neglect the Coulomb interaction between the charges. If a static, uniform, external electric field

$$\vec{E} = E_0 \hat{e}_z$$

is applied to this system, the equilibrium position is determined by

$$\mu\ddot{z} = -\mu\omega_0^2 z + qE_0 \stackrel{!}{=} 0,$$

leading to

$$z_0 = \frac{qE_0}{\mu\omega_0^2}.$$

The electric polarizability α_E is defined via the relation between the induced electric dipole moment and the external field⁶

$$\vec{p} = qz_0\hat{e}_z \equiv 4\pi\alpha_E\vec{E}.$$

For the harmonically bound system, α_E is proportional to the inverse of the spring constant,

$$\alpha_E = \frac{q^2}{4\pi} \frac{1}{\mu\omega_0^2},$$

i.e., it is a measure of the stiffness or rigidity of the system [12]. The potential energy associated with the induced electric dipole moment reads

$$V = -2\pi\alpha_E\vec{E}^2 = -\frac{1}{2}\vec{p} \cdot \vec{E}, \quad (34)$$

where the factor $\frac{1}{2}$ results from the interaction of an induced rather than a permanent electric dipole moment with the external field. Similarly, the potential of an induced magnetic dipole, $\vec{m} = 4\pi\beta_M\vec{H}$, is given by

$$V = -2\pi\beta_M\vec{H}^2. \quad (35)$$

III. COMPTON SCATTERING IN QUANTUM FIELD THEORY

Now that we have discussed Compton scattering in the framework of *nonrelativistic* quantum mechanics, we will turn to a description in the context of quantum field theory. Generally, we will consider the case of the nucleon but will restrict ourselves to the pion whenever this allows for a (substantial) simplification without loss of generality. We will direct our attention to the influence of hadron structure on the description of electromagnetic processes. In particular, we will emphasize the power of Ward-Takahashi identities [24,25]. First of all, we will describe the simplest electromagnetic vertex, namely, the interaction of a single photon with a charged pion. Using the method of Gell-Mann and Goldberger [9], we will derive the low-energy and low-momentum behavior of the (virtual) Compton-scattering tensor based upon Lorentz invariance, gauge invariance, crossing symmetry, and the discrete symmetries. Finally, we will consider Compton scattering off a pion to illustrate why off-shell effects are not directly observable.

⁶The factor 4π results from our use of the Heaviside-Lorentz units instead of the Gaussian system.

A. Electromagnetic vertex of a charged pion

For the purpose of illustrating the power of symmetry considerations, we explicitly discuss the most general electromagnetic vertex of an off-shell pion. We will formally introduce the concept of *form functions* by parameterizing the electromagnetic three-point Green's function of a pion. In this context, we distinguish between *form factors* and *form functions*, the former representing observables, which is, in general, not true for form functions.

Let us define the three-point Green's function of two unrenormalized pion field operators $\pi^+(x)$ and $\pi^-(y)$ and the electromagnetic current operator $J^\mu(z)$ as⁷

$$G^\mu(x, y, z) = \langle 0 | T [\pi^+(x) \pi^-(y) J^\mu(z)] | 0 \rangle, \quad (36)$$

and consider the corresponding momentum-space Green's function

$$(2\pi)^4 \delta^4(p_f - p_i - q) G^\mu(p_f, p_i) = \int d^4x d^4y d^4z e^{i(p_f \cdot x - p_i \cdot y - q \cdot z)} G^\mu(x, y, z), \quad (37)$$

where p_i and p_f are the four-momenta corresponding to the pion lines entering and leaving the vertex, respectively, and $q = p_f - p_i$ is the momentum transfer at the vertex. Defining the renormalized three-point Green's function G_R^μ as

$$G_R^\mu(p_f, p_i) = Z_\phi^{-1} Z_J^{-1} G^\mu(p_f, p_i), \quad (38)$$

where Z_ϕ and Z_J are renormalization constants,⁸ we obtain the one-particle irreducible, renormalized three-point Green's function by removing the propagators at the external lines,

$$\Gamma_R^{\mu, irr}(p_f, p_i) = [i\Delta_R(p_f)]^{-1} G_R^\mu(p_f, p_i) [i\Delta_R(p_i)]^{-1}, \quad (39)$$

where $\Delta_R(p)$ is the full, renormalized propagator. From a perturbative point of view, $\Gamma_R^{\mu, irr}$ is made up of those Feynman diagrams which cannot be disconnected by cutting any one single internal line.

In the following we will discuss a few model-independent properties of $\Gamma_R^{\mu, irr}(p_f, p_i)$.

1. Imposing Lorentz covariance, the most general parameterization of $\Gamma_R^{\mu, irr}$ can be written in terms of two independent four-momenta, $P^\mu = p_f^\mu + p_i^\mu$ and $q^\mu = p_f^\mu - p_i^\mu$, respectively, multiplied by Lorentz-scalar form functions F and G depending on three scalars, *e.g.*, q^2 , p_i^2 , and p_f^2 ,

$$\Gamma_R^{\mu, irr}(p_f, p_i) = (p_f + p_i)^\mu F(q^2, p_f^2, p_i^2) + (p_f - p_i)^\mu G(q^2, p_f^2, p_i^2). \quad (40)$$

⁷ $\pi^{+/-}(x)$ destroys a $\pi^{+/-}$ or creates a $\pi^{-/+}$.

⁸In fact, $Z_J = 1$ due to gauge invariance.

2. Time-reversal symmetry results in

$$F(q^2, p_f^2, p_i^2) = F(q^2, p_i^2, p_f^2), \quad G(q^2, p_f^2, p_i^2) = -G(q^2, p_i^2, p_f^2). \quad (41)$$

In particular, from Eq. (41) we conclude that $G(q^2, M_\pi^2, M_\pi^2) = 0$. This, of course, corresponds to the well-known fact that a spin-0 particle has only one electromagnetic form factor, $F(q^2)$.

3. Using the charge-conjugation properties $J^\mu \mapsto -J^\mu$ and $\pi^+ \leftrightarrow \pi^-$, it is straightforward to see that form functions of particles are just the negative of form functions of antiparticles. In particular, the π^0 does not have any electromagnetic form functions even off shell, since it is its own antiparticle.
4. Due to the hermiticity of the electromagnetic current operator, $F(q^2)$ is real in the spacelike region $q^2 \leq 0$:

$$\begin{aligned} (p_f + p_i)^\mu F^*(q^2) &= \langle p_f | J^\mu(0) | p_i \rangle^* = \langle p_i | J^{\mu\dagger}(0) | p_f \rangle = \langle p_i | J^\mu(0) | p_f \rangle \\ &= (p_i + p_f)^\mu F(q^2) \text{ for } q^2 \leq 0. \end{aligned}$$

5. After writing out the various time orderings in Eq. (36), let us consider the divergence

$$\begin{aligned} \partial_\mu^z G^\mu(x, y, z) &= \langle 0 | T[\pi^+(x) \pi^-(y) \partial_\mu J^\mu(z)] | 0 \rangle \\ &\quad + \delta(z^0 - x^0) \langle 0 | T\{[J^0(z), \pi^+(x)] \pi^-(y)\} | 0 \rangle \\ &\quad + \delta(z^0 - y^0) \langle 0 | T\{\pi^+(x) [J^0(z), \pi^-(y)]\} | 0 \rangle. \end{aligned} \quad (42)$$

Current conservation at the operator level, $\partial_\mu J^\mu(z) = 0$, together with the equal-time commutation relations of the electromagnetic charge-density operator with the pion field operators,⁹

$$\begin{aligned} [J^0(x), \pi^-(y)] \delta(x^0 - y^0) &= \delta^4(x - y) \pi^-(y), \\ [J^0(x), \pi^+(y)] \delta(x^0 - y^0) &= -\delta^4(x - y) \pi^+(y), \end{aligned} \quad (43)$$

are the basic ingredients for obtaining Ward-Takahashi identities [24,25] for electromagnetic processes. For example, we obtain from Eq. (42)

$$\partial_\mu^z G^\mu(x, y, z) = [\delta^4(z - y) - \delta^4(z - x)] \langle 0 | T[\pi^+(x) \pi^-(y)] | 0 \rangle. \quad (44)$$

Taking the Fourier transformation of Eq. (44), performing a partial integration, and repeating the same steps which lead from Eq. (37) to (39), one obtains the celebrated Ward-Takahashi identity for the electromagnetic vertex

⁹ Note that both equations are related by taking the adjoint.

$$q_\mu \Gamma_R^{\mu,irr}(p_f, p_i) = \Delta_R^{-1}(p_f) - \Delta_R^{-1}(p_i). \quad (45)$$

In general, this technique can be applied to obtain Ward-Takahashi identities relating Green's functions which differ by insertions of the electromagnetic current operator.

Inserting the parameterization of the irreducible vertex, Eq. (40), into the Ward-Takahashi identity, Eq. (45), the form functions F and G are constrained to satisfy

$$(p_f^2 - p_i^2)F(q^2, p_f^2, p_i^2) + q^2 G(q^2, p_f^2, p_i^2) = \Delta_R^{-1}(p_f) - \Delta_R^{-1}(p_i). \quad (46)$$

From Eq. (46) it can be shown that, given a consistent calculation of F , the propagator of the particle, Δ_R , as well as the form function G are completely determined (see Appendix A of Ref. [26] for details). The Ward-Takahashi identity thus provides an important consistency check for microscopic calculations.

6. As the simplest example, one may consider a structureless “point pion”:

$$\Gamma^\mu(p_f, p_i) = (p_f + p_i)^\mu, \quad q_\mu \Gamma^\mu = p_f^2 - p_i^2 = (p_f^2 - m_\pi^2) - (p_i^2 - m_\pi^2),$$

$$i.e., F(q^2, p_f^2, p_i^2) = 1, G(q^2, p_f^2, p_i^2) = 0.$$

7. As was already pointed out in Ref. [27], use of

$$\Gamma^\mu(p_f, p_i) = (p_f + p_i)^\mu F(q^2)$$

leads to an inconsistency, since the left-hand side of the corresponding Ward-Takahashi identity depends on q^2 , whereas the right-hand side only depends on p_f^2 and p_i^2 .

The nucleon case is more complicated due to spin and the most general form of the irreducible, electromagnetic vertex can be expressed in terms of 12 operators and associated form functions. The interested reader is referred to Refs. [28,29].

Finally, it is important to emphasize that the off-shell behavior of form functions is representation dependent, *i.e.*, form functions are, in general, not observable. In the context of a Lagrangian formulation, this can be understood as a result of field transformations [30–33]. This does not render the previous discussion useless, rather the Ward-Takahashi identities provide important consistency relations between the building blocks of a quantum-field-theoretical description.

B. Low-energy theorem for the Compton-scattering tensor

The Compton-scattering tensor $V_{s_i s_f}^{\mu\nu}$ is defined through a Fourier transformation of the time-ordered product of two electromagnetic current operators evaluated between on-shell nucleon states:¹⁰

¹⁰In the following, we will consider the proton case.

$$(2\pi)^4 \delta^4(p_f + q' - p_i - q) V_{s_i s_f}^{\mu\nu}(p_f, q'; p_i, q) = \int d^4x d^4y e^{-iq \cdot x} e^{iq' \cdot y} \langle N(p_f, s_f) | T[J^\mu(x) J^\nu(y)] | N(p_i, s_i) \rangle. \quad (47)$$

The relation to the invariant amplitude of real Compton scattering (RCS),¹¹ is given by

$$\mathcal{M} = -e^2 \epsilon_\mu \epsilon_\nu'^* V_{s_i s_f}^{\mu\nu}(p_f, q'; p_i, q) \Big|_{q^2=q'^2=0}. \quad (48)$$

In RCS, $V_{s_i s_f}^{\mu\nu}$ can only be tested for a rather restricted range of variables q^μ and q'^μ and, furthermore, only the transverse components of $V_{s_i s_f}^{\mu\nu}$ are accessible. The expression “virtual Compton scattering” (VCS) refers to the situation, where one or both photons are virtual. We will primarily be concerned with the case $q^2 < 0$ and $q'^2 = 0$ which, *e.g.*, for the proton can be tested in the reaction $e^- p \rightarrow e^- p \gamma$.

In the following, we will discuss the low-energy and small-momentum behavior of the Compton-scattering tensor. Our discussion will closely follow the method of Gell-Mann and Goldberger [9,35]. Let us first recall the distinction between $V_{s_i s_f}^{\mu\nu}$ and $\Gamma^{\mu\nu}$,¹²

$$V_{s_i s_f}^{\mu\nu}(p_f, q'; p_i, q) = \bar{u}(p_f, s_f) \Gamma^{\mu\nu}(P, q', q) u(p_i, s_i),$$

where $P = p_f + p_i$. In $V_{s_i s_f}^{\mu\nu}$ the nucleon is assumed to be on shell, $p_i^2 = p_f^2 = M^2$, whereas $\Gamma^{\mu\nu}$ is defined for arbitrary p_i^2 and p_f^2 , *i.e.*, $\Gamma^{\mu\nu}$ is the analogue of $\Gamma_R^{\mu, irr}$ of Eq. (39). We divide the contributions to $\Gamma^{\mu\nu}$ into two classes, *A* and *B*,

$$\Gamma^{\mu\nu} = \Gamma_A^{\mu\nu} + \Gamma_B^{\mu\nu},$$

where class *A* consists of the s- and u-channel pole terms and class *B* contains all the other contributions. With this separation all terms which are irregular for $q^\mu \rightarrow 0$ (or $q'^\mu \rightarrow 0$) are contained in class *A*, whereas class *B* is regular in this limit. Strictly speaking, one also assumes that there are no massless particles in the theory which could make a low-energy expansion in terms of kinematical variables impossible [8]. The contribution due to t-channel exchanges, such as a π^0 , is taken to be part of class *B*.

We express class *A* in terms of the full renormalized propagator and the irreducible electromagnetic vertices,

$$\Gamma_A^{\mu\nu} = \Gamma^\nu(p_f, p_f + q') iS(p_i + q) \Gamma^\mu(p_i + q, p_i) + \Gamma^\mu(p_f, p_f - q) iS(p_i - q') \Gamma^\nu(p_i - q', p_i). \quad (49)$$

Note that $\Gamma_A^{\mu\nu}$ is symmetric under photon crossing, $q \leftrightarrow -q'$ and $\mu \leftrightarrow \nu$, *i.e.*, $\Gamma_A^{\mu\nu}(P, q, q') = \Gamma_A^{\nu\mu}(P, -q', -q)$. Since this is also the case for the total $\Gamma^{\mu\nu}$, class *B* must be separately crossing symmetric [9]. In analogy to the previous section, Ward-Takahashi identities can be obtained for Γ^μ and $\Gamma^{\mu\nu}$,

¹¹We use the convention of Bjorken and Drell [34].

¹²We omit the superscript *irr* and the subscript *R*, respectively.

$$q_\mu \Gamma^\mu(p_f, p_i) = S^{-1}(p_f) - S^{-1}(p_i), \quad (50)$$

$$q_\mu \Gamma^{\mu\nu}(P, q', q) = i \left[S^{-1}(p_f) S(p_f - q) \Gamma^\nu(p_f - q, p_i) - \Gamma^\nu(p_f, p_i + q) S(p_i + q) S^{-1}(p_i) \right]. \quad (51)$$

Using Eq. (50), one obtains the following constraint for class A as imposed by gauge invariance:

$$q_\mu \Gamma_A^{\mu\nu}(P, q, q') = i \left[\Gamma^\nu(p_f, p_f + q') - \Gamma^\nu(p_i - q', p_i) + S^{-1}(p_f) S(p_i - q') \Gamma^\nu(p_i - q', p_i) - \Gamma^\nu(p_f, p_f + q') S(p_i + q) S^{-1}(p_i) \right]. \quad (52)$$

Eqs. (51) and (52) can now be combined to obtain a constraint for class B

$$q_\mu \Gamma_B^{\mu\nu} = q_\mu (\Gamma^{\mu\nu} - \Gamma_A^{\mu\nu}) = i [\Gamma^\nu(p_i - q', p_i) - \Gamma^\nu(p_f, p_f + q')]. \quad (53)$$

At this point, we make use of the freedom of choosing a convenient representation for Γ^μ below the pion production threshold,

$$\Gamma_{\text{eff}}^\mu(p_f, p_i) = \gamma^\mu F_1(q^2) + i \frac{\sigma^{\mu\nu} q_\nu}{2M} F_2(q^2) + q^\mu \not{q} \frac{1 - F_1(q^2)}{q^2}, \quad (54)$$

where F_1 and F_2 are the Dirac and Pauli form factors of the proton, respectively. The fundamental reason for this assumption is the fact that one can perform transformations of the fields in an effective Lagrangian which do not change the physical observables but which allow to a certain extent to transform away the off-shell dependence at the vertices. We will come back to this point in the next section.

Eq. (54) contains on-shell quantities only, and satisfies the Ward-Takahashi identity in combination with the free Feynman propagator,

$$q_\mu \Gamma_{\text{eff}}^\mu = \not{q} = S_F^{-1}(p_f) - S_F^{-1}(p_i).$$

In this representation the constraint for class B is particularly simple:

$$q_\mu \Gamma_B^{\mu\nu} = 0. \quad (55)$$

In order to solve this equation, one first makes an ansatz for class B ,

$$\Gamma_B^{\mu\nu}(P, q', q) = a^{\mu,\nu}(P, q') + b^{\mu\rho,\nu}(P, q') q_\rho + \dots \quad (56)$$

which is inserted into Eq. (55),

$$0 = a^{\mu,\nu}(P, q') q_\mu + b^{\mu\rho,\nu}(P, q') q_\mu q_\rho + \dots \quad (57)$$

The constraints due to crossing symmetry, and the discrete symmetries are imposed and Eq. (57) is solved as a power series in q and q' . At this point, off-shell kinematics is required in order to be able to treat q , q' , and P as completely independent. For example, using off-shell kinematics 6 invariant scalars can be formed from q , q' , and P , whereas for the on-shell case, $p_i^2 = p_f^2 = M^2$, only four of these combinations are independent. A detailed description of this procedure can be found in Ref. [36] and we will only summarize the key

results. Class B contains no constant terms and no terms at $\mathcal{O}(q)$ or $\mathcal{O}(q')$. Combining this observation with an evaluation of class A generates, for RCS, the LET of Refs. [8,9]. At $\mathcal{O}(2)$ one finds two terms which can be related to the electromagnetic polarizabilities $\bar{\alpha}_E$ and $\bar{\beta}_M$. The framework is general enough to also account for virtual photons. In particular, no new polarizability appears in the longitudinal part. In fact, this result has also been obtained in the framework of effective Lagrangians in Ref. [15].

The result for the RCS amplitude can be summarized as

$$\begin{aligned} \mathcal{M} = & -ie^2 \bar{u}(p_f, s_f) [\epsilon'^* \cdot \Gamma(-q') S_F(p_i + q) \epsilon \cdot \Gamma(q) + \epsilon \cdot \Gamma(q) S_F(p_i - q') \epsilon'^* \cdot \Gamma(-q') \\ & - \frac{4\pi}{e^2} \bar{\beta}_M (\epsilon \cdot \epsilon'^* q \cdot q' - \epsilon \cdot q' \epsilon'^* \cdot q) + \frac{\pi}{e^2 M^2} (\bar{\alpha}_E + \bar{\beta}_M) (\epsilon \cdot \epsilon'^* P \cdot q P \cdot q' + \epsilon \cdot P \epsilon'^* \cdot P q \cdot q' \\ & - \epsilon \cdot P \epsilon'^* \cdot q P \cdot q' - \epsilon \cdot q' \epsilon'^* \cdot P P \cdot q) + \mathcal{O}(3)] u(p_i, s_i), \end{aligned} \quad (58)$$

where we introduced the abbreviation

$$\Gamma^\mu(q) = \gamma^\mu + i \frac{\sigma^{\mu\nu} q_\nu}{2M} \kappa, \quad \kappa = 1.79. \quad (59)$$

Here, the electromagnetic polarizabilities are defined with respect to “Born terms” which have been calculated with the vertices of Eqs. (54) or (59) for RCS. In particular, with such a choice the Born terms are separately gauge invariant. As a matter of fact, this is not always the case, since, in principle, one could have started to calculate the Born terms with on-shell equivalent electromagnetic vertices containing the Sachs form factors G_E and G_M or the Barnes form factors H_1 and H_2 . Then class B would have taken a different form even though the final result for the total amplitude, of course, has to be the same. For a more detailed discussion of the ambiguity of defining “Born terms,” see Sec. IV of Ref. [36] as well as Ref. [37].

Table II contains a selection of results of various models for the electromagnetic polarizabilities which have to be compared with the empirical numbers of Tables III and IV. Within the framework of an effective Lagrangian it was shown in Ref. [15] that, in a covariant approach, the Compton polarizabilities $\bar{\alpha}_E$ and $\bar{\beta}_M$ coincide with the parameters determining the energy shifts in Eqs. (34) and (35). This should be compared with a nonrelativistic treatment, where, say, in the quadratic Stark effect only the second term of Eq. (31) appears in the energy shift. Whenever comparing different results, the original references should be consulted in order to see whether the predictions have been obtained in a nonrelativistic or a covariant framework.

The sum of the electric and magnetic polarizabilities is constrained by the Baldin sum rule [58],

$$(\bar{\alpha}_E + \bar{\beta}_M)_N = \frac{1}{2\pi^2} \int_{\omega_{thr}}^{\infty} \frac{\sigma_N^{\text{tot}}(\omega)}{\omega^2} d\omega, \quad (60)$$

where $\sigma_N^{\text{tot}}(\omega)$ is the total photoabsorption cross section. Eq. (60) is obtained via a once-subtracted dispersion relation for the spin-averaged forward Compton amplitude using the optical theorem together with the LET. An evaluation of the integral requires an extrapolation of available data to infinity (the results are given in units of 10^{-4} fm^3),

$$\begin{aligned}
(\bar{\alpha}_E + \bar{\beta}_M)_p &= 14.2 \pm 0.3, \text{ [59]} \\
(\bar{\alpha}_E + \bar{\beta}_M)_n &= 15.8 \pm 0.5, \text{ [60]} \\
(\bar{\alpha}_E + \bar{\beta}_M)_p &= 13.69 \pm 0.14, \text{ [61]} \\
(\bar{\alpha}_E + \bar{\beta}_M)_n &= 14.40 \pm 0.66, \text{ [61]}
\end{aligned} \tag{61}$$

where the last two results correspond to the most recent analysis.

Finally, we mention that four spin polarizabilities γ_i parameterize the amplitude at $\mathcal{O}(3)$ [62]. These spin-dependent terms have recently received considerable attention but a discussion of these structure constants is beyond the scope of the present treatment and we refer the interested reader to Refs. [63–66].

C. Compton scattering and off-shell effects

The issue of how to treat particles with “internal” structure as soon as they do not satisfy on-mass-shell kinematics has a long history. As an example, we have seen in Sec. III.A that the electromagnetic vertex of a pion involving off-mass-shell momenta is more complicated than for asymptotically free states. It is therefore natural to ask how such off-shell effects show up in observables and, in particular, whether they can be extracted from empirical information. Several attempts have been made to calculate off-shell effects within microscopic models and to estimate their importance in physical observables (see, *e.g.*, Refs. [28,29,67–69]).

We will argue in this section that off-shell effects are not only model dependent but also representation dependent and thus not directly measurable. In studying off-shell effects, we find that nucleon spin is an inessential complication. We use Compton scattering off a pion in the framework of chiral perturbation theory (ChPT) only as a *vehicle* to illustrate the point we want to make. Our conclusions are more general, *i.e.*, apply to other processes as well, and do not rely on chiral symmetry. We choose ChPT, since it provides a complete and consistent field-theoretical framework.

1. The chiral Lagrangian and field redefinitions

In this section we shall give a brief introduction to those aspects of chiral perturbation theory [70–72] which are relevant for a discussion of off-shell Green’s functions. We will introduce the concept of field transformations since it turns out to be important for interpreting the meaning of form functions.

In the limit of massless u , d , and s quarks, the QCD Lagrangian exhibits a chiral $SU(3)_L \times SU(3)_R$ symmetry which is assumed to be spontaneously broken to a subgroup isomorphic to $SU(3)_V$, giving rise to eight massless pseudoscalar Goldstone bosons with vanishing interactions in the limit of zero energies. In ChPT the chiral symmetry is mapped onto the most general effective Lagrangian for the interaction of these Goldstone bosons. The corresponding Lagrangian is organized in a momentum expansion [71–73],

$$\mathcal{L}_{\text{eff}} = \mathcal{L}_2 + \mathcal{L}_4 + \mathcal{L}_6 + \cdots, \tag{62}$$

where the index $2n$ denotes $2n$ derivatives. Couplings to external fields, such as the electromagnetic field, as well as explicit symmetry breaking due to the finite quark masses, can be systematically incorporated into the effective Lagrangian. Weinberg's power counting scheme allows for a classification of Feynman diagrams by establishing a relation between the momentum expansion and the loop expansion. Thus, a perturbative scheme is set up in terms of external momenta which are small compared to some scale. Covariant derivatives and quark-mass terms are counted as $\mathcal{O}(p)$ and $\mathcal{O}(p^2)$, respectively, in the power counting scheme.

The most general chiral Lagrangian at $\mathcal{O}(p^2)$ is given by

$$\mathcal{L}_2 = \frac{F_0^2}{4} \text{Tr} \left[D_\mu U (D^\mu U)^\dagger + \chi U^\dagger + U \chi^\dagger \right], \quad U(x) = \exp \left(i \frac{\phi(x)}{F_0} \right), \quad (63)$$

where

$$\phi(x) = \begin{pmatrix} \pi^0 + \frac{1}{\sqrt{3}}\eta & \sqrt{2}\pi^+ & \sqrt{2}K^+ \\ \sqrt{2}\pi^- & -\pi^0 + \frac{1}{\sqrt{3}}\eta & \sqrt{2}K^0 \\ \sqrt{2}K^- & \sqrt{2}\bar{K}^0 & -\frac{2}{\sqrt{3}}\eta \end{pmatrix}. \quad (64)$$

The quark-mass matrix is contained in $\chi = 2B_0 \text{diag}(m_u, m_d, m_s)$. B_0 is related to the quark condensate $\langle \bar{q}q \rangle$, $F_0 \approx 93$ MeV denotes the pion-decay constant in the chiral limit. The covariant derivative $D_\mu U = \partial_\mu U + ieA_\mu [Q, U]$, where $Q = \text{diag}(2/3, -1/3, -1/3)$ is the quark-charge matrix, $e > 0$, generates a coupling to the electromagnetic field A_μ . Finally, the equation of motion (EOM) obtained from \mathcal{L}_2 reads

$$\mathcal{O}_{EOM}^{(2)}(U) = D^2 U U^\dagger - U (D^2 U)^\dagger - \chi U^\dagger + U \chi^\dagger + \frac{1}{3} \text{Tr} (\chi U^\dagger - U \chi^\dagger) = 0. \quad (65)$$

The most general structure of \mathcal{L}_4 was first written down by Gasser and Leutwyler (see Eq. (6.16) of Ref. [72]),

$$\mathcal{L}_4 = L_1 \left\{ \text{Tr} [D_\mu U (D^\mu U)^\dagger] \right\}^2 + \dots, \quad (66)$$

and introduces 10 physically relevant low-energy coupling constants L_i .

We now discuss the concept of field transformations [30–33] by introducing a field redefinition,

$$U' = \exp(iS)U = U + iSU + \dots, \quad (67)$$

where $S = S^\dagger$ and $\text{Tr}(S) = 0$, and then look for generators S which i) are of $\mathcal{O}(p^2)$, ii) guarantee that U' has the correct $\text{SU}(3)_L \times \text{SU}(3)_R$ transformation properties, iii) produce the correct parity and charge-conjugation behavior, $P : U'(\vec{x}, t) \mapsto U'^\dagger(-\vec{x}, t)$, $C : U' \mapsto U'^T$. After some algebra (see Ref. [33] for details) one finds two such generators at $\mathcal{O}(p^2)$,

$$S = i\alpha_1 [D^2 U U^\dagger - U (D^2 U)^\dagger] + i\alpha_2 [\chi U^\dagger - U \chi^\dagger - \frac{1}{3} \text{Tr} (\chi U^\dagger - U \chi^\dagger)], \quad (68)$$

where α_1 and α_2 are arbitrary real parameters with dimension energy^{-2} . If we insert U' into \mathcal{L}_{eff} of Eq. (62), we obtain

$$\mathcal{L}_{\text{eff}}(U) \mapsto \mathcal{L}_{\text{eff}}(U') = \mathcal{L}_2(U) + \Delta\mathcal{L}_2(U) + \mathcal{L}_4(U) + \mathcal{O}(p^6), \quad (69)$$

where, to leading order in S , $\Delta\mathcal{L}_2(U)$ is given by

$$\Delta\mathcal{L}_2(U) = \text{total divergence} + \frac{F_0^2}{4} \text{Tr}(iS\mathcal{O}_{EOM}^{(2)}) + \mathcal{O}(p^6). \quad (70)$$

As usual, the total divergence is irrelevant. The second term of Eq. (70) is of $\mathcal{O}(p^4)$ and leads to a “modification” of \mathcal{L}_4 [26,74],

$$\mathcal{L}_4^{\text{off shell}} = \beta_1 \text{Tr}(\mathcal{O}_{EOM}^{(2)} \mathcal{O}_{EOM}^{(2)\dagger}) + \beta_2 \text{Tr}[(\chi U^\dagger - U \chi^\dagger) \mathcal{O}_{EOM}^{(2)}], \quad (71)$$

where $\alpha_1 = 4\beta_1/F_0^2$ and $\alpha_2 = -4(\beta_1 + \beta_2)/F_0^2$ and β_1 and β_2 are now dimensionless.

By a simple redefinition of the field variables one generates an infinite set of “new” Lagrangians depending on two parameters β_1 and β_2 . That all these Lagrangians describe the same physics will be illustrated in the next section. In this sense we would argue that Eqs. (62) and (69) represent the *same* theory in different representations. The concept of field transformations is very similar to choosing appropriate coordinates in the description of a dynamical system. The value of physical observables should, of course, not depend on the choice of coordinates.

2. The Compton-scattering amplitude

The most general, irreducible, renormalized three-point Green’s function [see Eq. (40)] at $\mathcal{O}(p^4)$ was derived in Ref. [26]. For positively charged pions and for real photons ($q^2 = 0$, $q = p_f - p_i$) it has the simple form

$$\Gamma_R^{\mu,irr}(p_f, p_i) = (p_f + p_i)^\mu \left(1 + 16\beta_1 \frac{p_f^2 + p_i^2 - 2M_\pi^2}{F_\pi^2} \right), \quad (72)$$

and the corresponding renormalized propagator satisfying the Ward-Takahashi identity, Eq. (45), is given by

$$i\Delta_R(p) = \frac{i}{p^2 - M_\pi^2 + \frac{16\beta_1}{F_\pi^2}(p^2 - M_\pi^2)^2 + i\epsilon}. \quad (73)$$

Clearly, the parameter β_1 is related to the deviation from a “pointlike” vertex, once one of the pion legs is off shell. Eqs. (72) and (73) have to be compared with the result of the usual representation of ChPT at $\mathcal{O}(p^4)$. In this case the vertex at $q^2 = 0$ is independent of p_f^2 and p_i^2 , $\Gamma_R^{\mu,irr}(p_f, p_i) = (p_f + p_i)^\mu$. Furthermore, the renormalized propagator is simply given by the free propagator.

Let us now consider the process $\gamma(\epsilon, q) + \pi^+(p_i) \rightarrow \gamma(\epsilon', q') + \pi^+(p_f)$. For $\beta_1 \neq 0$ one expects a different contribution of the pole terms, since the intermediate pion is not on its mass shell. We subtract the ordinary calculation of the pole terms using free vertices from the corresponding calculation with off-shell vertices and interpret the result as being due to off-shell effects. Similar methods have been the basis of investigating the influence

of off-shell form functions in various reactions involving the nucleon, such as proton-proton bremsstrahlung [67,69] or electron-nucleus scattering [29,68]. With the help of Eqs. (72) and (73) the change in the pole amplitude can easily be calculated,¹³

$$\Delta M_P = M_P(\beta_1 \neq 0) - M_P(\beta_1 = 0) = -ie^2 \frac{64\beta_1}{F_\pi^2} (p_f \cdot \epsilon' p_i \cdot \epsilon + p_f \cdot \epsilon p_i \cdot \epsilon'). \quad (74)$$

However, Eq. (74) cannot be used for a unique extraction of the form functions from experimental data since the very same term in the Lagrangian which contributes to the off-shell electromagnetic vertex also generates a two-photon contact interaction. This can be seen by inserting the appropriate covariant derivative into Eq. (71) and by selecting those terms which contain two powers of the pion field as well as two powers of the electromagnetic field. From the first term of Eq. (71) one obtains the following $\gamma\gamma\pi\pi$ interaction term

$$\begin{aligned} \Delta \mathcal{L}_{\gamma\gamma\pi\pi} = & \frac{16\beta_1 e^2}{F_\pi^2} \left\{ -A^2 [\pi^-(\square + M_\pi^2)\pi^+ + \pi^+(\square + M_\pi^2)\pi^-] \right. \\ & \left. + (\partial \cdot A + 2A \cdot \partial)\pi^+(\partial \cdot A + 2A \cdot \partial)\pi^- \right\}. \end{aligned} \quad (75)$$

For real photons Eq. (75) translates into a contact contribution of the form

$$\Delta \mathcal{M}_{\gamma\gamma\pi\pi} = ie^2 \frac{64\beta_1}{F_\pi^2} (p_f \cdot \epsilon' p_i \cdot \epsilon + p_f \cdot \epsilon p_i \cdot \epsilon'), \quad (76)$$

which precisely cancels the contribution of Eq. (74). At first sight the second term of Eq. (71) also seems to generate a contribution to the Compton-scattering amplitude. However, after wave function renormalization this term drops out (see Ref. [74] for details). We emphasize that all the cancellations happen only when one consistently calculates off-shell form functions, propagators and contact terms, and properly takes renormalization into account. Thus the Lagrangian of Eq. (69) which represents an equivalent form to the standard Lagrangian of ChPT yields, as a consequence of the equivalence theorem, the same Compton-scattering amplitude while, at the same time, it generates different off-shell form functions. Clearly, this illustrates why there is no unambiguous way of extracting the off-shell behavior of form functions from on-shell matrix elements. The ultimate reason is that the form functions of Eq. (40) are not only model dependent but also representation dependent, *i.e.*, two representations of the same theory result in the same observables but different form functions.

3. Off-shell effects versus contact interactions

In the present case there was a complete cancellation between off-shell effects and a contact contribution. Even though this will not always necessarily be true, we would still argue

¹³Of course, using Coulomb gauge $\epsilon^\mu = (0, \vec{\epsilon})$, $\epsilon'^\mu = (0, \vec{\epsilon}')$, and performing the calculation in the lab frame ($p_i^\mu = (M_\pi, 0)$), the additional contribution vanishes, since $p_i \cdot \epsilon = p_i \cdot \epsilon' = 0$. However, this is a gauge-dependent statement and thus not true for a general gauge.

that as a matter of principle it is impossible to *uniquely* extract off-shell effects from observables as there is a certain amount of freedom to trade such effects for contact interactions. Let us discuss this claim within a somewhat different approach which does not make use of a calculation within a specific model or theory. Such a discussion also serves to demonstrate that our interpretation is independent of the fact that we made use of ChPT at $\mathcal{O}(p^4)$. For that purpose we come back to the method of Gell-Mann and Goldberger in their derivation of the low-energy theorem for Compton scattering [9], and split the most general invariant amplitude into two classes A and B (see Sec. III.B), where class A consists of the most general pole terms and class B contains the rest. The original motivation in Ref. [9] for such a separation was to isolate those terms of $\mathcal{M} = -ie^2\epsilon_\mu\epsilon_\nu^*M^{\mu\nu}$ which have a singular behavior in the limit $q, q' \rightarrow 0$. As in Eq. (49) we write class A in terms of the most general expressions for the irreducible, renormalized vertices and the renormalized propagator,

$$M_A^{\mu\nu} = \Gamma^\nu(p_f, p_f + q')\Delta_R(p_i + q)\Gamma^\mu(p_i + q, p_i) + (q \leftrightarrow -q', \mu \leftrightarrow \nu), \quad (77)$$

where we made use of crossing symmetry. For sufficiently low energies class B can be expanded in terms of the relevant kinematical variables [see Eq. (56)]. Furthermore, in class A we expand the vertices and the renormalized pion propagator around their respective on-shell points, $p^2 = M_\pi^2$. We obtain for the propagator

$$\Delta_R^{-1}(p) = p^2 - M_\pi^2 - \Sigma(p^2) = (p^2 - M_\pi^2)[1 - \frac{p^2 - M_\pi^2}{2}\Sigma''(M_\pi^2) + \dots], \quad (78)$$

where we made use of the standard normalization conditions $\Sigma(M_\pi^2) = \Sigma'(M_\pi^2) = 0$. The expansion of, *e.g.*, the vertex describing the absorption of the initial real photon in the s channel reads

$$\begin{aligned} \Gamma^\mu(p_i + q, p_i) &= (P^\mu + q'^\mu)F[0, M_\pi^2 + (s - M_\pi^2), M_\pi^2] \\ &= (P^\mu + q'^\mu)[1 + (s - M_\pi^2)\partial_2 F(0, M_\pi^2, M_\pi^2) + \dots], \end{aligned} \quad (79)$$

where $P = p_i + p_f$, and where ∂_2 refers to partial differentiation with respect to the second argument. Inserting the result of Eqs. (78) and (79) into Eq. (77) we obtain for the s -channel contribution to $M_A^{\mu\nu}$

$$\begin{aligned} M_s^{\mu\nu} &= -ie^2(P^\nu + q'^\nu)[1 + (s - M_\pi^2)\partial_3 F(0, M_\pi^2, M_\pi^2) + \dots] \\ &\quad \times \frac{1}{s - M_\pi^2}[1 + \frac{s - M_\pi^2}{2}\Sigma''(M_\pi^2) + \dots] \\ &\quad \times [P^\mu + q'^\mu](1 + (s - M_\pi^2)\partial_2 F(0, M_\pi^2, M_\pi^2) + \dots] \\ &= -ie^2 \frac{(P^\nu + q'^\nu)(P^\mu + q'^\mu)}{s - M_\pi^2} + \mathcal{O}[(s - M_\pi^2)^0] \\ &= \text{“free” } s \text{ channel} + \text{analytical terms}, \end{aligned} \quad (80)$$

and an analogous term for the u channel. In Eq. (80) “free” s channel refers to a calculation with on-shell vertices. From Eq. (80) we immediately see that off-shell effects resulting from either the form functions or the renormalized propagator are of the same order as analytical contributions from class B . In the total amplitude off-shell contributions from class A cannot

uniquely be separated from class B contributions. In the language of field transformations this means that contributions to \mathcal{M} can be shifted between different diagrams leaving the total result invariant. In the language of Gell-Mann and Goldberger, by a change of representation, contributions can be shifted from class A to class B within the *same* theory. We can also express this differently; what appears to be an off-shell effect in one representation results, for example, from a contact interaction in another representation. In this sense, off-shell effects are not only model dependent, *i.e.*, different models generate different off-shell form functions, but they are also representation dependent which means that even different representations of the *same* theory generate different off-shell form functions. This has to be contrasted with on-shell S-matrix elements which, in general, will be different for different models (model dependent), but always the same for different representations of the same model (representation independent). For a further discussion in the context of bremsstrahlung the reader is referred to Ref. [75].

IV. VIRTUAL COMPTON SCATTERING

In this section we will, finally, address an old topic [76] which has recently received considerable attention, namely, low-energy virtual Compton scattering (VCS) as tested in, *e.g.*, the reactions $e^-p \rightarrow e^-p\gamma$ [77–81] and $\pi^-e^- \rightarrow \pi^-e^-\gamma$ [82,83]. From a theoretical point of view, the objective of investigating VCS is to map out all independent components of the Compton-scattering tensor, Eq. (47), for arbitrary four-momenta of the photons. As we have seen, real Compton scattering is only sensitive to the transverse components as well as restricted to the kinematics $\omega = |\vec{q}|$ and $\omega' = |\vec{q}'|$. As in all studies with electromagnetic probes, the possibilities to investigate the structure of the target increase substantially, if virtual photons are used since (a) photon energy and momentum can be varied independently and (b) longitudinal components of the transition current are accessible.

A. Kinematics and LET

When investigating $e^-p \rightarrow e^-p\gamma$ or $\pi^-e^- \rightarrow \pi^-e^-\gamma$, the VCS tensor is only a building block of the invariant amplitude describing the process. The total amplitude consists of a Bethe-Heitler (BH) piece, where the real photon is emitted by the initial or final electrons, and the VCS contribution (see Fig. 2),

$$\mathcal{M} = \mathcal{M}_{\text{BH}} + \mathcal{M}_{\text{VCS}}. \quad (81)$$

It is straightforward to calculate the Bethe-Heitler contribution, since it involves on-shell information of the target only, namely, its electromagnetic form factors. In the following we will be concerned with the invariant amplitude for VCS. For the final photon the Lorentz condition $q' \cdot \epsilon' = 0$ is automatically satisfied, and we choose, in addition, the Coulomb gauge $\epsilon'^\mu = (0, \vec{\epsilon}')$ implying $\vec{q}' \cdot \vec{\epsilon}' = 0$. Writing the invariant amplitude as

$$\mathcal{M}_{\text{VCS}} = -ie^2 \epsilon_\mu M^\mu,$$

where $\epsilon_\mu = e\bar{u}_e\gamma_\mu u_e/q^2$ is the polarization vector of the virtual photon, we can make use of current conservation, $q_\mu\epsilon^\mu = 0$, $q_\mu M^\mu = 0$, to express ϵ_0 and M^0 in terms of ϵ_z and M_z , respectively,

$$\mathcal{M}_{\text{VCS}} = ie^2 \left(\vec{\epsilon}_T \cdot \vec{M}_T + \frac{q^2}{\omega^2} \epsilon_z M_z \right). \quad (82)$$

Note that as $\omega \rightarrow 0$, both ϵ_z and M_z tend to zero such that \mathcal{M}_{VCS} in Eq. (82) remains finite.

After a reduction from Dirac spinors to Pauli spinors the transverse and longitudinal parts of Eq. (82) may be expressed in terms of 8 and 4 independent structures, respectively [10,15,36] (see Table VII):

$$\begin{aligned} \vec{\epsilon}_T \cdot \vec{M}_T &= \vec{\epsilon}'^* \cdot \vec{\epsilon}_T A_1 + i\vec{\sigma} \cdot (\vec{\epsilon}'^* \times \vec{\epsilon}_T) A_2 + (\hat{q}' \times \vec{\epsilon}'^*) \cdot (\hat{q} \times \vec{\epsilon}_T) A_3 + i\vec{\sigma} \cdot (\hat{q}' \times \vec{\epsilon}'^*) \times (\hat{q} \times \vec{\epsilon}_T) A_4 \\ &\quad + i\hat{q} \cdot \vec{\epsilon}'^* \vec{\sigma} \cdot (\hat{q} \times \vec{\epsilon}_T) A_5 + i\hat{q}' \cdot \vec{\epsilon}_T \vec{\sigma} \cdot (\hat{q}' \times \vec{\epsilon}'^*) A_6 + i\hat{q} \cdot \vec{\epsilon}'^* \vec{\sigma} \cdot (\hat{q}' \times \vec{\epsilon}_T) A_7 \\ &\quad + i\hat{q}' \cdot \vec{\epsilon}_T \vec{\sigma} \cdot (\hat{q} \times \vec{\epsilon}'^*) A_8, \end{aligned} \quad (83)$$

$$\epsilon_z M_z = \epsilon_z \vec{\epsilon}'^* \cdot \hat{q} A_9 + i\epsilon_z \vec{\epsilon}'^* \cdot \hat{q} \vec{\sigma} \cdot (\hat{q}' \times \hat{q}) A_{10} + i\epsilon_z \vec{\sigma} \cdot (\hat{q} \times \vec{\epsilon}'^*) A_{11} + i\epsilon_z \vec{\sigma} \cdot (\hat{q}' \times \vec{\epsilon}'^*) A_{12}, \quad (84)$$

where the functions A_i depend on three kinematical variables, *e.g.*, $|\vec{q}|$, $\omega' = |\vec{q}'|$, and $z = \hat{q} \cdot \hat{q}'$. For the spin-zero case only one longitudinal and two transverse structures are required. In Ref. [36] the method of Gell-Mann and Goldberger was extended to VCS (see Sec. III.B) and model-independent predictions for the functions A_i were obtained to second order in q or q' . The results for the functions A_i in the CM system expanded up to $\mathcal{O}(2)$, *i.e.* $|\vec{q}'|^2$, $|\vec{q}'||\vec{q}|$ and $|\vec{q}|^2$, are shown in Tables V and VI. To this order, all A_i can be expressed in terms of the proton mass M , the anomalous magnetic moment κ , the electromagnetic Sachs form factors G_E and G_M , the mean square electric radius r_E^2 , and the real-Compton-scattering electromagnetic polarizabilities $\bar{\alpha}_E$ and $\bar{\beta}_M$. For $|\vec{q}| = \omega'$, the predictions of Table V reduce to the well-known RCS result.

In Ref. [37] the low-energy behavior of the VCS amplitude of $\pi^-(p_i) + \gamma^*(\epsilon, q) \rightarrow \pi^-(p_f) + \gamma(\epsilon', q')$ was found to be of the form

$$\mathcal{M}_{\text{VCS}} = -2ie^2 F(q^2) \left[\frac{p_f \cdot \epsilon'^* (2p_i + q) \cdot \epsilon}{s - m_\pi^2} + \frac{p_i \cdot \epsilon'^* (2p_f - q) \cdot \epsilon}{u - m_\pi^2} - \epsilon \cdot \epsilon'^* \right] + \mathcal{M}_R, \quad (85)$$

where $F(q^2)$ is the electromagnetic form factor of the pion, $s = (p_i + q)^2$, and $u = (p_i - q')^2$. The residual term \mathcal{M}_R is separately gauge invariant and at least of second order, *i.e.*, $\mathcal{O}(qq, qq', q'q')$.

B. Beyond the LET: Generalized Polarizabilities

The framework for analyzing the model-dependent terms specific to VCS at low energies has been developed by Guichon, Liu, and Thomas [10]. The invariant amplitude \mathcal{M}_{VCS} is split into a pole piece \mathcal{M}_P and a residual part \mathcal{M}_R . For the nucleon, the s- and u-channel pole diagrams have been calculated using the free Feynman propagator together with electromagnetic vertices of the form

$$\Gamma_{F_1 F_2}^\mu(p_f, p_i) = \gamma^\mu F_1(q^2) + i \frac{\sigma^{\mu\nu} q_\nu}{2M} F_2(q^2), \quad q = p_f - p_i, \quad (86)$$

where F_1 and F_2 are the Dirac and Pauli form factors, respectively. The corresponding amplitude $\mathcal{M}_P^{\gamma^* \gamma}$ contains all irregular terms as $q \rightarrow 0$ or $q' \rightarrow 0$ and is separately gauge invariant [10,36]. The second property is a special feature when working with these particular vertex operators and has been essential for the derivation in [10]. Gauge invariance with this choice of operators is not as trivial as it might appear, since the electromagnetic vertex, Eq. (86), and the nucleon propagator do not satisfy the Ward-Takahashi identity (except at the real photon point, $q^2 = 0$):

$$q_\mu \Gamma_{F_1, F_2}^\mu(p_f, p_i) = (\not{p}_f - \not{p}_i) F_1(q^2) \neq S_F^{-1}(p_f) - S_F^{-1}(p_i). \quad (87)$$

However, explicit calculation, including the use of the Dirac equation, shows that \mathcal{M}_P of Ref. [10] is, in fact, identical with evaluating the pole terms using the vertex of Eq. (54). We should mention that in Ref. [10] the phrase LET is used for the Born terms as opposed to the more restrictive sense of Sec. IV.A.

For the pion, the situation is somewhat more complicated due to the fact that even for real photons the s- and u-channel pole diagrams are not separately gauge invariant. A natural starting point is given by Eq. (85).

The generalized polarizabilities in VCS [10] result from an analysis of $\mathcal{M}_R^{\gamma^* \gamma}$ in terms of electromagnetic multipoles $H^{(\rho' L', \rho L)S}(\omega', |\vec{q}|)$, where ρ (ρ') denotes the type of the initial (final) photon ($\rho = 0$: charge, C; $\rho = 1$: magnetic, M; $\rho = 2$: electric, E). The initial (final) orbital angular momentum is denoted by L (L'), and S distinguishes between non-spin-flip ($S = 0$) and spin-flip ($S = 1$) transitions. For the pion, only the case $S = 0$ applies. $\mathcal{M}_R^{\gamma^* \gamma}$ is at least linear in the energy of the real photon. A restriction to the lowest-order, *i.e.* linear terms in ω' leads to only electric and magnetic dipole radiation in the final state. Parity and angular-momentum selection rules (see Table VIII) then allow for 3 scalar multipoles ($S = 0$) and 7 vector multipoles ($S = 1$). The corresponding ten GPs, $P^{(01,01)0}, \dots, \hat{P}^{(11,2)1}$, are functions of $|\vec{q}|^2$, where mixed-type polarizabilities, $\hat{P}^{(\rho' L', L)S}(|\vec{q}|^2)$, have been introduced which are neither purely electric nor purely Coulomb type (see Ref. [10] for details).

However, the treatment of Ref. [10] does not fully exploit all the symmetries of the VCS tensor, resulting in redundant structures. This observation was first made in Ref. [85] in the covariant framework of the linear sigma model. In fact, only six of the above ten GPs are independent, if charge-conjugation symmetry is imposed in combination with particle crossing [86,87]. For example, for a charged pion, the constraint for the Compton tensor reads [37,86]

$$M_{\pi^+}^{\mu\nu}(p_f, q'; p_i, q) \stackrel{C}{=} M_{\pi^-}^{\mu\nu}(p_f, q'; p_i, q) \stackrel{part. cross.}{=} M_{\pi^+}^{\mu\nu}(-p_i, q'; -p_f, q), \quad (88)$$

generating one relation between originally three GPs [85,86]:

$$0 = \sqrt{\frac{3}{2}} P^{(01,01)0}(|\vec{q}|^2) + \sqrt{\frac{3}{8}} P^{(11,11)0}(|\vec{q}|^2) + \frac{3|\vec{q}|^2}{2\omega_0} \hat{P}^{(01,1)0}(|\vec{q}|^2), \quad (89)$$

allowing one to eliminate the mixed-type polarizability $\hat{P}^{(01,1)0}$. In Eq. (89), $\omega_0 = \omega|_{\omega'=0} = M - \sqrt{M^2 + \vec{q}^2}$. The remaining two spin-independent polarizabilities have been defined such as to be proportional to the RCS polarizabilities at $|\vec{q}| = 0$:

$$\bar{\alpha}_E = -\frac{e^2}{4\pi} \sqrt{\frac{3}{2}} P^{(01,01)0}(0), \quad \bar{\beta}_M = -\frac{e^2}{4\pi} \sqrt{\frac{3}{8}} P^{(11,11)0}(0). \quad (90)$$

Note that a calculation in the framework of nonrelativistic quantum mechanics is not particle-crossing symmetric and fails to satisfy the second equality in Eq. (88) [88] (see Sec. II.B). In general, we do not expect a nonrelativistic calculation to reproduce the constraints of [86,87].

Relations between the spin-dependent GPs at $|\vec{q}| = 0$ and the four spin-dependent RCS polarizabilities γ_i of Ref. [62] were discussed in Ref. [89]:

$$\gamma_3 = -\frac{e^2}{4\pi} \frac{3}{\sqrt{2}} P^{(01,12)1}(0), \quad \gamma_2 + \gamma_4 = -\frac{e^2}{4\pi} \frac{3\sqrt{3}}{2\sqrt{2}} P^{(11,02)1}(0), \quad (91)$$

i.e., only two of the four γ_i can be related to GPs at $|\vec{q}| = 0$.

C. Generalized Polarizabilities of the Nucleon

Predictions for the generalized polarizabilities of the nucleon have been obtained in various approaches [10,85,90–97]. Here, we will concentrate on a discussion of the GPs obtained within the heavy-baryon formulation of chiral perturbation theory (HBChPT) and the linear sigma model. Both calculations are based upon covariant approaches and thus satisfy the constraints found in [86,87].

The extension of meson chiral perturbation theory to the nucleon sector starts from the most general effective chiral Lagrangian involving nucleons, pions, and external fields [98],

$$\mathcal{L}_{\text{eff}} = \mathcal{L}_{\pi N}^{(1)} + \mathcal{L}_{\pi N}^{(2)} + \dots \quad (92)$$

The lowest-order Lagrangian,

$$\mathcal{L}_{\pi N}^{(1)} = \bar{\Psi} \left(i \not{D} - m_0 + \frac{\overset{\circ}{g}_A}{2} \not{u} \gamma_5 \right) \Psi, \quad (93)$$

contains two parameters in the chiral limit, namely, the axial-vector coupling constant $\overset{\circ}{g}_A$ and the nucleon mass m_0 . The covariant derivatives $D_\mu \Psi$ includes, among other terms, the coupling to the electromagnetic field, and u_μ contains in addition the derivative coupling of a pion. Since the nucleon mass does not vanish in the chiral limit, one has an additional large scale such that even in the chiral limit external four-momenta cannot be made arbitrarily small. In the framework of HBChPT [44,99] four-momenta are written as $p = m_0 v + k$, $v^2 = 1$, $v^0 \geq 1$, where often $v^\mu = (1, 0, 0, 0)$ is used. By introducing so-called velocity-dependent fields

$$\mathcal{N}_v = e^{+im_0 v \cdot x} \frac{1}{2} (1 + \not{v}) \Psi, \quad \mathcal{H}_v = e^{+im_0 v \cdot x} \frac{1}{2} (1 - \not{v}) \Psi,$$

such that $\Psi(x) = e^{-im_0 v \cdot x} (\mathcal{N}_v + \mathcal{H}_v)$, and by using the equation of motion for \mathcal{H}_v , one can eliminate \mathcal{H}_v to obtain a Lagrangian for \mathcal{N}_v which, to lowest order in $1/m_0$, is given by

$$\hat{\mathcal{L}}_{\pi N}^{(1)} = \bar{N}_v (i v \cdot D + g_A S \cdot u) \mathcal{N}_v. \quad (94)$$

In Eq. (94) $S^\mu = i\gamma_5 \sigma^{\mu\nu} v_\nu$ refers to the spin matrix. This procedure can be generalized to higher orders in the chiral expansion and is very similar to the Foldy-Wouthuysen procedure [100]. At each order in the momentum expansion one will have $1/m_0$ terms coming from the expansion of the leading Lagrangian in combination with counter terms resulting from the most general form allowed at that order.

In Refs. [92,93] the VCS amplitude was calculated using HBChPT to third order in the external momenta. At $\mathcal{O}(p^3)$, contributions to the GPs are generated by nine one-loop diagrams and the π^0 -exchange t -channel pole graph (see Fig. 2 of Ref. [92]). For the loop diagrams only the leading-order Lagrangians of Eqs. (63) and (94) are needed. For the π^0 -exchange diagram one requires, in addition, the $\pi^0 \gamma \gamma^*$ vertex provided by the Wess-Zumino-Witten Lagrangian [102,103],

$$\mathcal{L}_{\gamma\gamma\pi^0}^{(WZW)} = -\frac{e^2}{32\pi^2 F_\pi} \epsilon^{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta} \pi^0, \quad (95)$$

where $\epsilon_{0123} = 1$ and $F_{\mu\nu}$ is the electromagnetic field strength tensor. At $\mathcal{O}(p^3)$, the LET of VCS (see Tables V and VI) is reproduced by the tree-level diagrams obtained from Eq. (94) and the relevant part of the second- and third-order Lagrangian [104],

$$\hat{\mathcal{L}}_{\pi N}^{(2)} = -\frac{1}{2M} \bar{N}_v \left[D \cdot D + \frac{e}{2} (\mu_S + \tau_3 \mu_V) \epsilon_{\mu\nu\rho\sigma} F^{\mu\nu} v^\rho S^\sigma \right] N_v, \quad (96)$$

$$\hat{\mathcal{L}}_{\pi N}^{(3)} = \frac{ie\epsilon_{\mu\nu\rho\sigma} F^{\mu\nu}}{8M^2} \bar{N}_v \left[\mu_S - \frac{1}{2} + \tau_3 (\mu_V - \frac{1}{2}) \right] S^\rho D^\sigma N_v + h.c.. \quad (97)$$

The numerical results for the ten GPs are shown in Fig. 3 (recall that only six combinations are independent). As an example of the spin-independent sector, let us discuss the generalized electric polarizability $\bar{\alpha}_E(|\vec{q}|^2) = -\frac{e^2}{4\pi} \sqrt{\frac{3}{2}} P^{(01,01)0}(|\vec{q}|^2)$,

$$\frac{\bar{\alpha}_E(\vec{q}^2)}{\bar{\alpha}_E} = 1 - \frac{7}{50} \frac{|\vec{q}|^2}{m_\pi^2} + \frac{81}{2800} \frac{|\vec{q}|^4}{m_\pi^4} + O\left(\frac{|\vec{q}|^6}{m_\pi^6}\right), \quad \bar{\alpha}_E = \frac{5e^2 g_A^2}{384\pi^2 m_\pi F_\pi^2} = 12.8 \times 10^{-4} \text{ fm}^3. \quad (98)$$

As a function of $|\vec{q}|^2$, the prediction of ChPT decreases considerably faster than within the constituent quark model [10]. At $\mathcal{O}(p^3)$, all results can be expressed in terms of the pion mass m_π , the axial-vector coupling constant g_A , and the pion-decay constant F_π . This property is true for all GPs at $\mathcal{O}(p^3)$.

The π^0 -exchange diagrams only contributes to the spin-dependent polarizabilities. Let us consider as an example $P^{(11,11)1}$,

$$P^{(11,11)1}(|\vec{q}|^2) = -\frac{1}{288} \frac{g_A^2}{F_\pi^2} \frac{1}{\pi^2 M} \left[\frac{|\vec{q}|^2}{m_\pi^2} - \frac{1}{10} \frac{|\vec{q}|^4}{m_\pi^4} \right] + \frac{1}{3M} \frac{g_A}{8\pi^2 F_\pi^2} \tau_3 \left[\frac{|\vec{q}|^2}{m_\pi^2} - \frac{|\vec{q}|^4}{m_\pi^4} \right] + O\left(\frac{|\vec{q}|^6}{m_\pi^6}\right). \quad (99)$$

In general, the spin-dependent polarizabilities consist of an isoscalar contribution due to pion loops [first term of Eq. (99)] and an isovector contribution from the π^0 -exchange graph [second term of Eq. (99)].

The linear sigma model (LSM) [105] represents a field-theoretical realization of chiral $SU(2)_L \times SU(2)_R$ symmetry. The dynamical degrees of freedom are given by a nucleon doublet Ψ , a pion triplet $\vec{\pi}$, and a singlet σ :

$$\mathcal{L}_S = i\bar{\Psi}\not{\partial}\Psi + \frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma + \frac{1}{2}\partial_\mu\vec{\pi}\cdot\partial^\mu\vec{\pi} - g_{\pi N}\bar{\Psi}(\sigma + i\gamma_5\vec{\tau}\cdot\vec{\pi})\Psi - \frac{\mu^2}{2}(\sigma^2 + \vec{\pi}^2) - \frac{\lambda}{4}(\sigma^2 + \vec{\pi}^2)^2, \quad (100)$$

$$\mathcal{L}_{s.b.} = -c\sigma, \quad (101)$$

where $\mathcal{L}_{s.b.}$ is a perturbation which explicitly breaks chiral symmetry. With an appropriate choice of parameters ($\mu^2 < 0$, $\lambda > 0$) the model reveals spontaneous symmetry breaking, $\langle 0|\sigma|0\rangle = F_\pi = 92.4$ MeV. The spectrum consists of massless pions, a massive sigma and nucleons with masses satisfying the Goldberger-Treiman relation $m_N = g_{\pi N}F_\pi$ with $g_A = 1$. The symmetry breaking of Eq. (101) generates the PCAC relation

$$\partial^\mu A_\mu^a = F_\pi m_\pi^2 \pi^a.$$

The interaction with the electromagnetic field is introduced via minimal substitution in Eq. (100). The generalized polarizabilities have been calculated in the framework of a one-loop calculation [85,94]. In Fig. 4, some predictions of the LSM are shown in comparison with other models. In Ref. [94] for each generalized polarizability a chiral expansion has been performed. As an example, consider $P^{(11,11)1}$ for the proton,

$$P_p^{(11,11)1}(Q_0^2) = Q_0^2 \frac{C}{2m_\pi} \left[\frac{6}{\mu} - \frac{1}{2\mu} + \frac{7\pi}{8} + \mathcal{O}(\mu) \right] + (Q_0^2)^2 \frac{C}{5m_\pi^3} \left[-\frac{15}{\mu} + \frac{1}{8\mu} - \frac{9\pi}{64} + \mathcal{O}(\mu) \right] + \mathcal{O}[(Q_0^2)^3], \quad (102)$$

where $C = g_{\pi N}^2/(72\pi^2 m_N^4)$, $\mu = m_\pi/m_N$ and $Q_0^2 = 2m_N(\sqrt{|\vec{q}|^2 + m_N^2} - m_N)$. The leading-order term of the chiral expansion agrees with the corresponding result of HBChPT at $\mathcal{O}(p^3)$.

D. Generalized Polarizabilities of Pions

Of course, the concept of (generalized) polarizabilities can also be applied to the pion. From the experimental point of view, an extraction of polarizabilities is more complicated since there is no pion target. For that purpose, one has to consider reactions which contain the Compton-scattering amplitude as a building block, such as, *e.g.*, the Primakoff effect in high-energy pion-nucleon bremsstrahlung, $\pi^- A \rightarrow A\pi^- \gamma$ [106,107], radiative pion photo-production off the nucleon, $\gamma p \rightarrow n\gamma\pi^+$ [108], and pion pair production in e^+e^- scattering, $e^+e^- \rightarrow e^+e^-\pi^+\pi^-$ [109,110]. The current empirical information on the RCS polarizabilities is summarized in Table IX. New experiments are presently being performed [111] or in preparation [112].

From a theoretical point of view, a precise determination of the electromagnetic polarizabilities is of great importance as a test of ChPT. At the one-loop level, $\mathcal{O}(p^4)$, of chiral

perturbation theory the electromagnetic polarizabilities of the charged pion are entirely determined by an $\mathcal{O}(p^4)$ counter term [12],

$$\bar{\alpha}_E = -\bar{\beta}_M = \frac{e^2}{4\pi m_\pi F_\pi^2} (2l_5^r - l_6^r) = (2.68 \pm 0.42) \times 10^{-4} \text{ fm}^3, \quad (103)$$

where the linear combination $2l_5^r - l_6^r$ is predicted through the decay $\pi^+ \rightarrow e^+ \nu_e \gamma$. Corrections to this result at $\mathcal{O}(p^6)$ were shown to be reasonably small, namely 12% and 24% of the $\mathcal{O}(p^4)$ values for $\bar{\alpha}_E$ and $\bar{\beta}_M$, respectively [113]. In particular, the degeneracy $|\bar{\alpha}_E| = |\bar{\beta}_M|$ of Eq. (103) is lifted at $\mathcal{O}(p^6)$.

Presently, the pion VCS reaction is under investigation as part of the Fermilab E781 SELEX experiment, where a 600 GeV pion beam interacts with atomic electrons in nuclear targets [83]. In principle, the different behavior under the substitution $\pi^- \rightarrow \pi^+$ of \mathcal{M}_{BH} and \mathcal{M}_{VCS} (see Fig. 2),

$$\mathcal{M}_{\text{BH}}(\pi^-) = -\mathcal{M}_{\text{BH}}(\pi^+), \quad \mathcal{M}_{\text{VCS}}(\pi^-) = \mathcal{M}_{\text{VCS}}(\pi^+), \quad (104)$$

may be of use in identifying the contributions due to internal structure by comparing the reactions involving a π^- and a π^+ beam for the same kinematics:¹⁴

$$d\sigma(\pi^+) - d\sigma(\pi^-) \sim 4Re \left(\mathcal{M}_{\text{BH}}(\pi^+) \mathcal{M}_{\text{VCS}}^*(\pi^+) \right). \quad (105)$$

The invariant amplitude for VCS at $\mathcal{O}(p^4)$ in the framework of chiral perturbation theory has been calculated in Ref. [101]. The result was found to be in complete agreement with Eq. (85). Using the procedure developed in Ref. [86], the generalized polarizabilities of the charged pion in the convention of Ref. [10] were extracted,

$$\begin{aligned} \bar{\alpha}_E(|\vec{q}|^2) &= -\bar{\beta}_M(|\vec{q}|^2) \\ &= \frac{e^2}{8\pi m_\pi} \sqrt{\frac{m_\pi}{E_\pi}} \left[\frac{4(2l_5^r - l_6^r)}{F_\pi^2} - 2 \frac{m_\pi - E_\pi}{m_\pi} \frac{1}{(4\pi F)^2} J^{(0)'} \left(2 \frac{m_\pi - E_\pi}{m_\pi} \right) \right], \end{aligned} \quad (106)$$

where $E_\pi = \sqrt{m_\pi^2 + |\vec{q}|^2}$ and

$$J^{(0)'}(x) = \frac{1}{x} \left[1 - \frac{2}{x\sigma(x)} \ln \left(\frac{\sigma(x) - 1}{\sigma(x) + 1} \right) \right], \quad \sigma(x) = \sqrt{1 - \frac{4}{x}}, \quad x \leq 0.$$

The $|\vec{q}|^2$ dependence does not contain any $\mathcal{O}(p^4)$ parameter, *i.e.*, it is entirely given in terms of the pion mass and the pion decay constant $F_\pi = 92.4$ MeV. The numerical prediction is shown in Fig. 5.

¹⁴This argument works for any particle which is not its own antiparticle such as the K^+ or K^0 . Of course, one could also employ the substitution $e^- \rightarrow e^+$.

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FIGURES

FIG. 1. Compton scattering: kinematics.

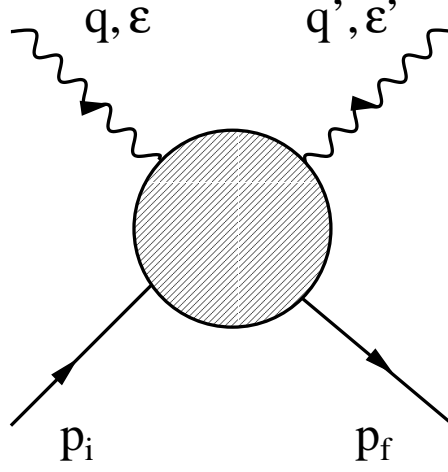


FIG. 2. Bethe-Heitler diagrams (a) and (b). VCS diagram (c). The four-momenta of the virtual photons in the BH diagrams and the VCS diagram differ from each other.

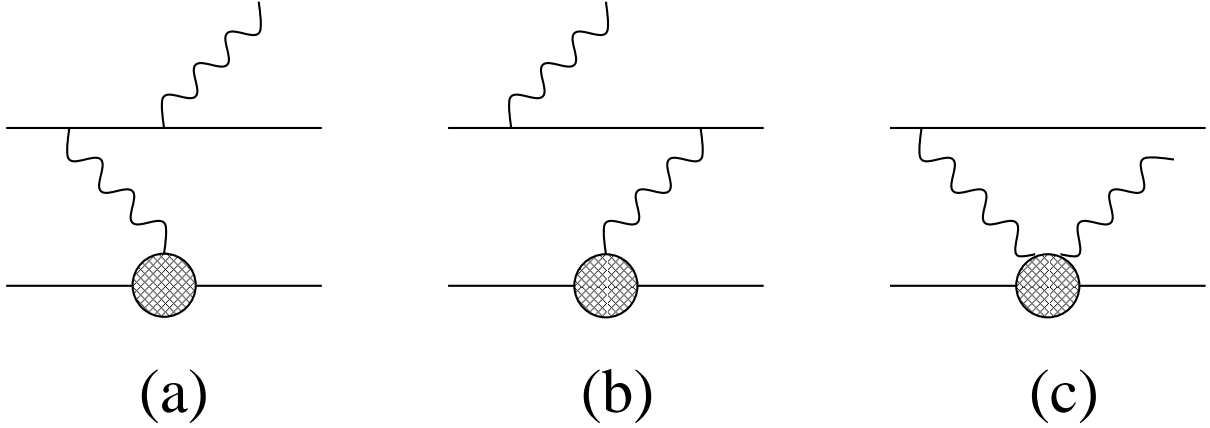


FIG. 3. $\mathcal{O}(p^3)$ prediction for the GPs of the proton as a function of $\vec{q}^2 = |\vec{q}|^2$ (from Ref. [93]). The dashed line represents the contribution from pion-nucleon loops, the dotted line represents the π^0 -exchange graph, and the dash-dotted line represents the sum of both.

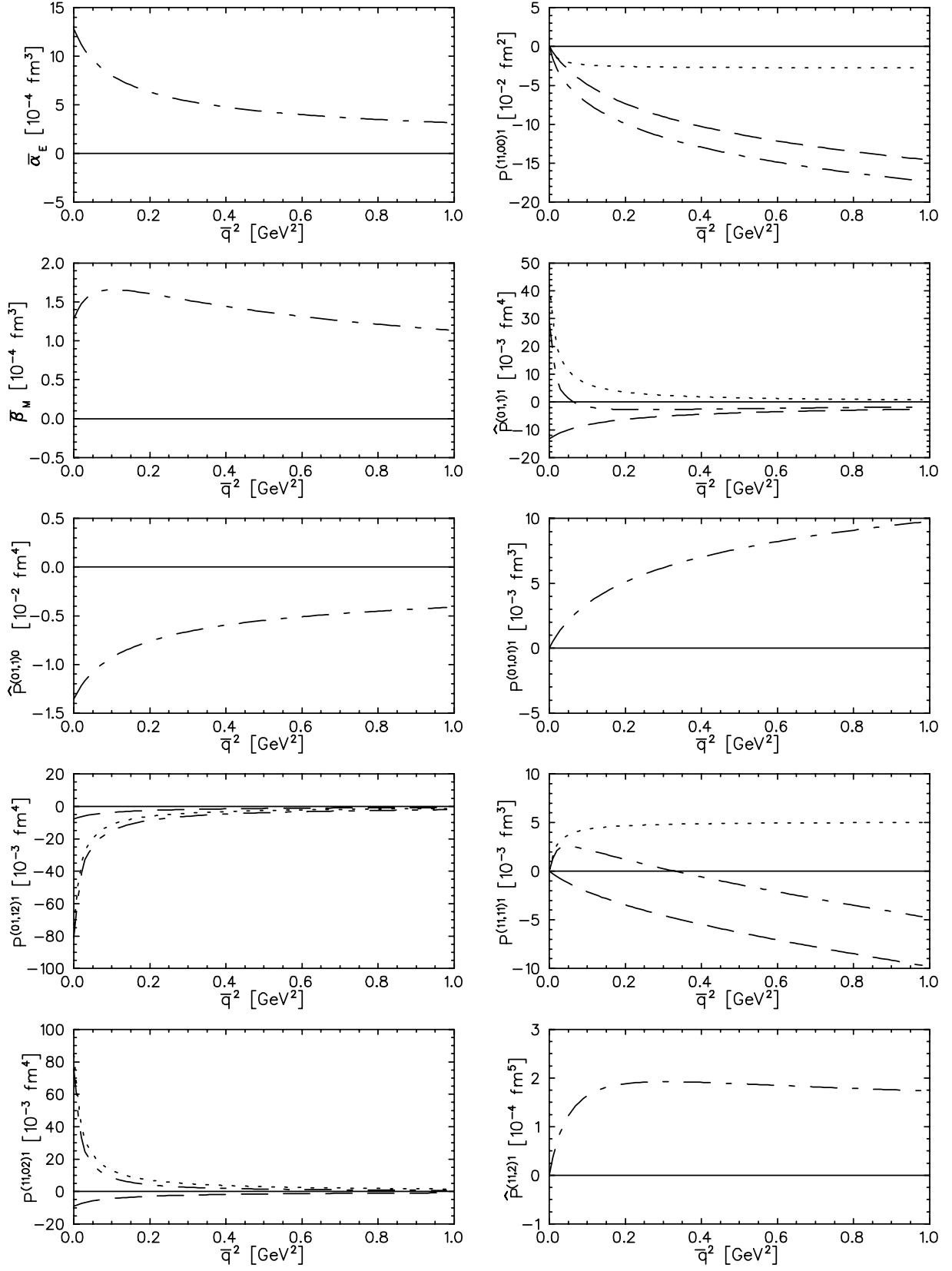


FIG. 4. Some GPs of the proton. $Q_0^2 = 2m_N(\sqrt{|\vec{q}|^2 + m_N^2} - m_N)$. Solid curve: linear sigma model [85]. Dashed curve: Constituent quark model [10]. Dotted curve: Effective Lagrangian model [91]. Dashed-dotted curve: Chiral perturbation theory [92].

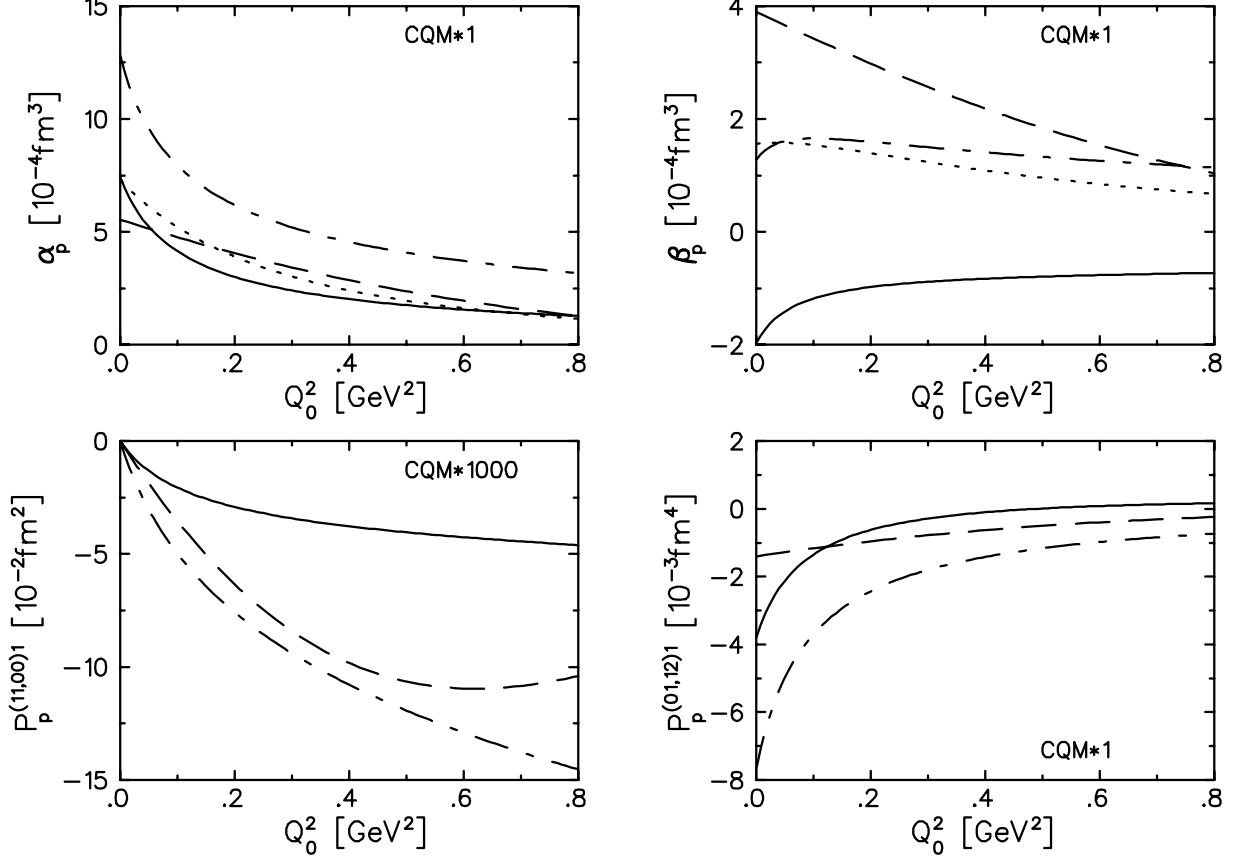
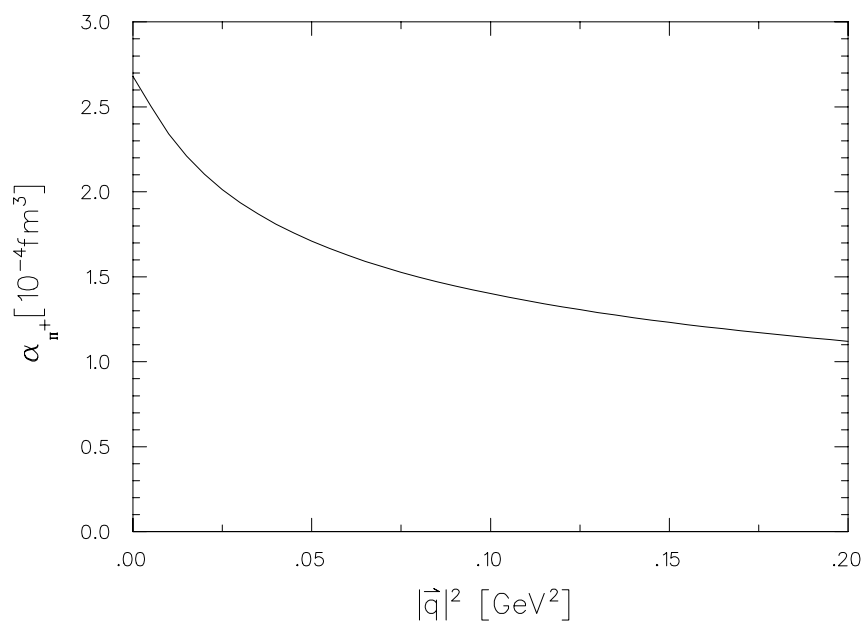


FIG. 5. Generalized polarizability $\bar{\alpha}_E(|\vec{q}|^2)$ of the charged pion in ChPT [101]. Note that $\bar{\alpha}_E(|\vec{q}|^2) = -\bar{\beta}_M(|\vec{q}|^2)$ at $\mathcal{O}(p^4)$.



TABLES

TABLE I. Thomson cross section σ_T for the electron, charged pion, and the proton.

| particle | σ_T |
|----------|-----------------|
| electron | 0.665 barn |
| pion | 8.84 μ barn |
| proton | 197 nbarn |

TABLE II. Some theoretical predictions for the electromagnetic polarizabilities of the nucleon in units of 10^{-4} fm^3 .

| Ref. | description | $\bar{\alpha}_E$ proton | $\bar{\beta}_M$ proton | $\bar{\alpha}_E$ neutron | $\bar{\beta}_M$ neutron |
|------|---------------------------|----------------------------|---------------------------|-----------------------------|----------------------------|
| [38] | MIT Bag | 7.1 | 2.6 | 4.7 | 3.4 |
| [39] | Skyrme model | | 2 | | 2 |
| [40] | MIT Bag | 10.8 | 2.3 | 10.8 | 1.5 |
| [41] | Chiral Quark Model | 7-9 | ≤ 2 | 7-9 | ≤ 2 |
| [42] | Skyrme | 8.3 25.2 | 8.5 1.7 | 8.3 25.2 | 8.5 1.7 |
| [43] | Chiral Soliton | 13.4 | -1.1 | 13.4 | -1.1 |
| [44] | HBChPT $\mathcal{O}(p^3)$ | 12.8 | 1.3 | 12.8 | 1.3 |
| [45] | NRQM | 7.25 | 12 | 7.25 | 12 |
| [46] | ϵ^3 ChPT | 17.1 | 9.2 | 17.1 | 9.2 |

TABLE III. Empirical numbers for the electromagnetic polarizabilities of the proton in units of 10^{-4} fm^3 . The polarizabilities have been determined using the Baldin sum rule, Eq. (60), with $\bar{\alpha}_p + \bar{\beta}_p = 14.2 \pm 0.5$. Due to this constraint the errors of $\bar{\alpha}_p$ and $\bar{\beta}_p$ are anticorrelated.

| Ref. | description | $\bar{\alpha}_E^p$ | $\bar{\beta}_M^p$ |
|------|---------------------------------|-----------------------------------|----------------------------------|
| [47] | $\gamma p \rightarrow \gamma p$ | $10.9 \pm 2.2 \pm 1.3$ | $3.3 \pm 2.2 \pm 1.3$ |
| [48] | $\gamma p \rightarrow \gamma p$ | $10.62^{+1.25+1.07}_{-1.19-1.03}$ | $3.58^{+1.19+1.03}_{-1.25-1.07}$ |
| [49] | $\gamma p \rightarrow \gamma p$ | $9.8 \pm 0.4 \pm 1.1$ | $4.4 \pm 0.4 \pm 1.1$ |
| [50] | $\gamma p \rightarrow \gamma p$ | $12.5 \pm 0.6 \pm 0.9$ | $1.7 \pm 0.6 \pm 0.9$ |
| [50] | global average | $12.1 \pm 0.8 \pm 0.5$ | $2.1 \pm 0.8 \pm 0.5$ |

TABLE IV. Empirical numbers for the electric polarizability of the neutron in units of 10^{-4} fm^3 .

| Ref. | description | $\bar{\alpha}_E^n$ |
|------|--|------------------------|
| [51] | low-energy $n\text{Pb}$ and $n\text{C}$ scattering | 12 ± 10 |
| [52] | low-energy $n\text{Pb}$ and $n\text{Bi}$ scattering | 8 ± 10 |
| [53] | quasi-free Compton scattering: $\gamma d \rightarrow \gamma' np$ | $11.7^{+4.3}_{-11.7}$ |
| [54] | quasi-free Compton scattering: $\gamma d \rightarrow \gamma' np$ | $10.7^{+3.3}_{-10.7}$ |
| [55] | low-energy $n\text{Pb}$ scattering | $12.0 \pm 1.5 \pm 2.0$ |
| [56] | low-energy $n\text{Pb}$ scattering | 0 ± 5 |
| [57] | PDG average | $9.8^{+1.9}_{-2.3}$ |

TABLE V. Transverse functions A_i of Eq. (83) in the CM frame. The functions are expanded in terms of $|\vec{q}'|$ and $|\vec{q}|$ of the final real and initial virtual photon, respectively. $N_i = \sqrt{\frac{E_i+M}{2M}}$ is the normalization factor of the initial spinor, where $E_i = \sqrt{M^2 + |\vec{q}|^2}$. $G_E(q^2) = F_1(q^2) + \frac{q^2}{4M^2} F_2(q^2)$ and $G_M(q^2) = F_1(q^2) + F_2(q^2)$ are the electric and magnetic Sachs form factors, respectively. $r_E^2 = 6G_E'(0) = (0.74 \pm 0.02) \text{ fm}^2$ is the electric mean square radius [84] and $\kappa = 1.79$ the anomalous magnetic moment of the proton. Q^μ is defined as $q^\mu|_{|\vec{q}'|=0} = (M - E_i, \vec{q})$, $Q^2 = -2M(E_i - M)$, and $z = \hat{q}' \cdot \hat{q}$. $\bar{\alpha}_E$ and $\bar{\beta}_M$ are the electric and magnetic Compton polarizabilities of the proton, respectively.

| | |
|-------|--|
| A_1 | $-\frac{1}{M} + \frac{z}{M^2} \vec{q} - \left(\frac{1}{8M^3} + \frac{r_E^2}{6M} - \frac{\kappa}{4M^3} - \frac{4\pi\bar{\alpha}_E}{e^2} \right) \vec{q}' ^2 + \left(\frac{1}{8M^3} + \frac{r_E^2}{6M} - \frac{z^2}{M^3} + \frac{(1+\kappa)\kappa}{4M^3} \right) \vec{q} ^2$ |
| A_2 | $\frac{1+2\kappa}{2M^2} \vec{q}' - \frac{\kappa^2}{4M^3} \vec{q}' ^2 + \frac{z\kappa}{2M^3} \vec{q}' \vec{q} - \frac{(1+\kappa)^2}{4M^3} \vec{q} ^2$ |
| A_3 | $-\frac{1}{M^2} \vec{q} + \left(\frac{1}{4M^3} + \frac{4\pi\bar{\beta}_M}{e^2} \right) \vec{q}' \vec{q} + \frac{(3-2\kappa-\kappa^2)z}{4M^3} \vec{q} ^2$ |
| A_4 | $-\frac{(1+\kappa)^2}{2M^2} \vec{q} - \frac{(2+\kappa)\kappa}{4M^3} \vec{q}' \vec{q} + \frac{(1+\kappa)^2 z}{4M^3} \vec{q} ^2$ |
| A_5 | $-\frac{N_i G_M(Q^2)}{(E_i+z \vec{q})(E_i+M)} \frac{ \vec{q} ^2}{ \vec{q}' } + \frac{(1+\kappa)\kappa}{4M^3} \vec{q} ^2$ |
| A_6 | $\frac{1+\kappa}{2M^2} \vec{q}' - \frac{(1+\kappa)\kappa}{4M^3} \vec{q}' ^2 - \frac{(1+\kappa)z}{2M^3} \vec{q}' \vec{q} $ |
| A_7 | $-\frac{1+3\kappa}{4M^3} \vec{q}' \vec{q} $ |
| A_8 | $\frac{1+3\kappa}{4M^3} \vec{q}' \vec{q} $ |

TABLE VI. Longitudinal functions A_i of Eq. (84) in the CM frame. See caption of Table V.

| | |
|----------|---|
| A_9 | $\frac{N_i G_E(Q^2)}{(E_i+z \vec{q})(E_i+M)} \frac{ \vec{q} ^2}{ \vec{q}' } - \frac{1}{M} + \frac{z}{M^2} \vec{q} - \left(\frac{1}{8M^3} + \frac{r_E^2}{6M} - \frac{\kappa}{4M^3} - \frac{4\pi\bar{\alpha}_E}{e^2} \right) \vec{q}' ^2 + \left(\frac{1}{8M^3} + \frac{r_E^2}{6M} - \frac{z^2}{M^3} \right) \vec{q} ^2$ |
| A_{10} | $-\frac{1+3\kappa}{4M^3} \vec{q}' \vec{q} $ |
| A_{11} | $-\frac{1+2\kappa}{2M^2} \vec{q}' + \frac{\kappa^2}{4M^3} \vec{q}' ^2 + \frac{(1+\kappa)z}{4M^3} \vec{q}' \vec{q} + \frac{1+2\kappa}{4M^3} \vec{q} ^2$ |
| A_{12} | $\frac{(1+\kappa)z}{2M^2} \vec{q}' - \frac{(1+\kappa)\kappa z}{4M^3} \vec{q}' ^2 - \frac{(1+\kappa)(2z^2-1)}{4M^3} \vec{q}' \vec{q} - \frac{(1+\kappa)z}{4M^3} \vec{q} ^2$ |

TABLE VII. Number of independent amplitudes for the description of the general VCS tensor.

| nucleon | RCS ($\gamma\gamma$) | VCS ($\gamma^*\gamma$) | VCS ($\gamma^*\gamma^*$) |
|----------------------------|------------------------|--------------------------|----------------------------|
| total number of amplitudes | 6 | 12 | 18 |
| spin independent (= pion) | 2 | 3 | 5 |
| spin dependent | 4 | 9 | 13 |

TABLE VIII. Multipolarities of initial and final states.

| J^P | final real photon | initial virtual photon |
|-----------------|-------------------|------------------------|
| $\frac{1}{2}^-$ | E1 | C1,E1 |
| $\frac{3}{2}^-$ | E1 | C1,E1,M2 |
| $\frac{1}{2}^+$ | M1 | C0,M1 |
| $\frac{3}{2}^+$ | M1 | C2,E2,M1 |

TABLE IX. Empirical numbers for the electromagnetic polarizabilities of the pion in units of 10^{-4} fm^3 .

| reaction | $\bar{\alpha}_E$ | β_M | $\bar{\alpha}_E + \beta_M$ |
|---------------------------------------|---------------------|--|---|
| $\pi^- A \rightarrow A \pi^- \gamma$ | 6.8 ± 1.4 [106] | $-7.1 \pm 2.8_{\text{stat.}} \pm 1.8_{\text{syst.}}$ [107] | $1.4 \pm 3.1_{\text{stat.}} \pm 2.5_{\text{syst.}}$ [107] |
| $\gamma p \rightarrow \gamma \pi^+ n$ | 20 ± 12 [108] | - | - |